The Chern-Simons path integral and the quantum Racah formula

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Abstract

We generalize several results on Chern-Simons models on $\Sigma \times S^1$ in the so-called "torus gauge" which were obtained recently in [32] (= arXiv:math-ph/0507040) to the case of general (simply-connected simple compact) structure groups and general link colorings. In particular, we give a non-perturbative evaluation of the Wilson loop observables corresponding to a special class of simple but non-trivial links and show that their values are given by Turaev's shadow invariant. As a byproduct we obtain a heuristic path integral derivation of the quantum Racah formula.

1 Introduction

In 1988 E. Witten succeeded in defining, on a physical level of rigor, a large class of new 3-manifold (link) invariants with the help of the heuristic Chern-Simons path integral, cf. [53]. Later a rigorous definition of these invariants was given, cf. [42, 41] and part I of [47]. The approach in [42, 41] is based on the representation theory of quantum groups[1] and uses surgery techniques on the base manifold. A related approach is the so-called "shadow world" approach by Turaev (cf. [48] and part II of [47]), which also works with quantum groups but replaces the use of surgery operations by certain combinatorial arguments leading to finite "state sums".

It is an open problem (cf., e.g., p. 2 in [25] and Problem (P1) in [32]) how the rigorous approaches using quantum groups are related to Witten's path integral approach. This problem should be interesting for the following two reasons:

Firstly, one can hope that the solution of this problem will lead to some new insights into (the representation theory of) quantum groups. Quantum groups are rather complicated algebraic objects and the corresponding representation theory (if the deformation parameter q is a root of unity) is highly non-trivial. The constructions in [47] rely on several rather deep algebraic results, cf. [5, 50] and Chap. XI, Sec. 6 in [47]. In contrast, Witten's path integral expressions are strikingly simple. Thus it is reasonable to expect that a better understanding of the relationship between the CS path integral approach and the two rigorous algebraic approaches in [42, 41, 47] will also lead to a better understanding of (the representation theory of) quantum groups.

Secondly, and more importantly, one can expect that the solution of the aforementioned problem will lead to some progress towards the solution of one of the central open problems in the field, namely the question if/how one can make rigorous sense of the path integral expressions used in the heuristic treatment in [53] (cf. Sec. 6 below for additional comments).

The results in [32], which were obtained by extending the work in [12, 13, 14, 31] in a suitable way, suggest that the key for establishing a direct relationship between the CS path integral and the two quantum group approaches mentioned above is the so-called "torus gauge fixing" procedure, introduced in [12] for the study of CS models on base manifolds M of the form $M = \Sigma \times S^1$. Indeed, already in [12] it was demonstrated that in the torus gauge setting the evaluation of the Wilson loop observables (WLOs) of special links consisting exclusively of "vertical loops" naturally leads to the S-matrix expressions on the right-hand side of the so-called fusion rules, cf. expression (13) below and Remark 5 in the Appendix. In [32] it was then shown how to treat the case of general links within (a suitably modified version of) the torus gauge setting. Moreover, it was shown that in the special case $G = SU(2)$ the evaluation of the Wilson loop observables of loops without double points naturally leads to the gleam factors and the summation over (admissible) "area colorings" present in Turaev's formula for the shadow invariant

[1] in fact, the approach in [42, 41, 47] is somewhat more general, cf. Remark 3 below

(cf. Eq. (25) below). In the present paper we will generalize the results in [32] to general (simply-connected simple compact) groups G and to links with arbitrary "colors", i.e. equipped with arbitrary representations (and not only the fundamental representation as in [32]). As a result we will be able to demonstrate that within the torus gauge setting also the fusion coefficients (i.e. the numbers N^i_{jl} in Eq. (21)) in Turaev's formula for the shadow invariant appear naturally when links without double points are studied.

We mention here that Turaev's shadow invariant also appears in the evaluation of a purely two-dimensional quantum field theory, namely q-deformed Yang-Mills theory on a Riemannian surface Σ [20]. The connection of the latter with Chern-Simons on S^1-bundles over Σ, of which $S^1 \times \Sigma$ is a special case, was developed in [19, 20, 1, 21, 22, 15]. The algebraic lattice formulation of q-deformed two-dimensional Yang-Mills has been worked out for real q and not for q being a root of unity [18]. Although we will not further develop the connection to this two-dimensional theory in this paper, we note that the intermediate expressions we obtain in our evaluation of the Chern-Simons path integral are those of q-deformed two-dimensional Yang-Mills. In turn, the path integral formulation of the simpler two-dimensional quantum field theory may be helpful in defining the Chern-Simons path integral on non-trivial bundles over Σ [15, 11].

The paper is organized as follows. In Subsec. 2.1 we first recall some important concepts and construction from Lie theory. In Subsec. 2.2 we then introduce (elementary versions of) the relevant concepts of the theory of affine Lie algebras (resp. Conformal Field theory) which played a role in [53]. In Sec. 3 we reformulate Turaev's shadow invariant for manifolds of the form $\Sigma \times S^1$ using the notation from Sec. 2. In Secs. 4.1–4.3 we recall some of the results obtained in [12, 13, 14, 31, 32] on Chern-Simons models on $\Sigma \times S^1$ in the "torus gauge" and in Subsec. 4.4 we then generalize the calculations in [32] for the WLOs for links without double points to the case of general (simple simply-connected compact) groups G and arbitrary link colorings. In Sec. 5 we show that the finite state sums appearing in Sec. 4 are equivalent to the state sums in the shadow invariant. In other words, the values of the WLOs obtained in Sec. 4 agree exactly with the values obtained by applying the shadow invariant to the corresponding links, cf. Eq. (74). Finally, after giving a brief outlook in Sec. 6, we show in the Appendix that by reversing the order of arguments used in Secs. 4 and 5 one can obtain a path integral derivation of the so-called quantum Racah formula (cf. Eq. (17) below).

2 Algebraic preliminaries

2.1 Concepts from classical Lie theory

Let G be a simply-connected and simple compact Lie group and $\mathfrak{g}$ its Lie algebra. Moreover, let T be a maximal torus of G and $\mathfrak{t}$ the Lie algebra of T. (We will keep G and T fixed for the rest of this paper).

- $(\cdot,\cdot)$ denotes the Killing metric on $\mathfrak{g}$ normalized such that $(\alpha,\alpha) = 2$ if α is a long root. In the sequel we will identify $\mathfrak{t}$ and $\mathfrak{t}^*$ with the help of $(\cdot,\cdot)$. We set $r := \dim(\mathfrak{t}) = \mathrm{rank}(\mathfrak{g})$. $\pi_{\mathfrak{t}} : \mathfrak{g} \to \mathfrak{t}$ will denote the $(\cdot,\cdot)$-orthogonal projection and $\mathfrak{t}^{\perp}$ the $(\cdot,\cdot)$-orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$.

- $\mathcal{R} \subset \mathfrak{t}^*$ will denote the set of roots associated to $(\mathfrak{g},\mathfrak{t})$ and $\check{\mathcal{R}}$ the set of inverse roots, i.e. $\check{\mathcal{R}}$ is given by $\check{\mathcal{R}} := \{\check{\alpha} \mid \alpha \in \mathcal{R}\} \subset \mathfrak{t}$ where $\check{\alpha} := \frac{2\alpha}{(\alpha,\alpha)}$. Let $\Lambda \subset \mathfrak{t}^*$ denote the weight lattice associated to $(\mathfrak{g},\mathfrak{t})$, i.e. Λ is given by
$$\Lambda := \{\lambda \in \mathfrak{t}^* \mid \lambda(\check{\alpha}) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{R}\} \tag{1}$$
$\Lambda_{\check{\mathcal{R}}}$ will denote the lattice generated by the inverse roots.

- A Weyl chamber is a connected component of $\mathfrak{t}\backslash \bigcup_{\alpha\in\mathcal{R}} H_\alpha$ where $H_\alpha := \alpha^{-1}(0)$. A Weyl alcove (or "affine Weyl chamber") is a connected component of the set[2] $\mathfrak{t}_{reg} := \mathfrak{t}\backslash \bigcup_{\alpha\in\mathcal{R},k\in\mathbb{Z}} H_{\alpha,k}$ where $H_{\alpha,k} := \alpha^{-1}(k)$.

- Let $\mathcal{W}$ denote the Weyl group (associated to $\mathfrak{g}$ and $\mathfrak{t}$), i.e. the group of isometries of $\mathfrak{t} \cong \mathfrak{t}^*$ generated by the orthogonal reflections on the hyperplanes H_α, $\alpha \in \mathcal{R}$, defined above. $\mathcal{W}_{\text{aff}}$ will denote the

[2] note that in [31] we used the notation $\mathfrak{t}'_{reg}$ instead of $\mathfrak{t}_{reg}$.

affine Weyl group, i.e. the group of isometries of $\mathfrak{t} \cong \mathfrak{t}^*$ generated by the orthogonal reflections on the hyperplanes $H_{\alpha,k}$, $\alpha \in \mathcal{R}$, $k \in \mathbb{Z}$, defined above[3]. For $\tau \in \mathcal{W}_{\text{aff}}$ we will denote the sign of τ by $\text{sgn}(\tau)$.

In the sequel let us fix a Weyl chamber $\mathcal{C}$. Let P denote the unique Weyl alcove which is contained in $\mathcal{C}$ and has $0 \in \mathfrak{t}$ on its boundary.

- Let $\mathcal{R}_+$ denote the set of positive roots, i.e. $\mathcal{R}_+ := \{\alpha \in \mathcal{R} \mid (\alpha, x) \geq 0 \text{ for all } x \in \overline{\mathcal{C}}\}$, and let Λ_+ denote the set of "dominant weights", i.e. $\Lambda_+ := \Lambda \cap \overline{\mathcal{C}}$.

- For $\lambda \in \Lambda_+$ let ρ_λ denote the (up to equivalence) unique irreducible complex representation of G with highest weight λ and χ_λ the character corresponding to ρ_λ. The multiplicity of the global weight associated to μ in χ_λ will be denoted by $m_\lambda(\mu)$, i.e. we have

$$\chi_\lambda(\exp(b)) = \sum_{\mu \in \Lambda} m_\lambda(\mu) e^{2\pi i (\mu, b)} \quad \text{for all } b \in \mathfrak{t} \tag{2}$$

- ρ will denote the half-sum of the positive roots and θ the unique long root in the Weyl chamber $\mathcal{C}$. The dual Coxeter number $c_{\mathfrak{g}}$ of $\mathfrak{g}$ is given by[4]

$$c_{\mathfrak{g}} = 1 + (\theta, \rho) \tag{3}$$

- For each $\lambda \in \Lambda_+$ we set

$$C_2(\lambda) := (\lambda, \lambda + 2\rho) \tag{4}$$

i.e., $C_2(\lambda)$ is the second Casimir element (w.r.t. to the inner product $(\cdot, \cdot)$) corresponding to the irreducible representation of $\mathfrak{g}$ with highest weight λ.

- For $\lambda \in \Lambda_+$ let $\overline{\lambda} \in \Lambda_+$ denote the weight conjugated to λ and $\lambda^* \in \Lambda_+$ the weight conjugated to λ "after applying a shift by ρ". More precisely, λ^* is given by $\lambda^* + \rho = \overline{\lambda + \rho}$.

Remark 1 Let $I \subset \mathfrak{t}$ denote the "integral lattice", i.e. $I := \ker(\exp_{|\mathfrak{t}})$. From the assumption that G is simply-connected it follows that I coincides with the lattice $\Lambda_{\check{\mathcal{R}}}$ generated by the inverse roots so the weight lattice Λ associated to $(\mathfrak{g}, \mathfrak{t})$ coincides with the weight lattice I^* of (G, T) given by $I^* := \{\alpha \in \mathfrak{t}^* \mid \alpha(x) \in \mathbb{Z} \text{ for all } x \in I\}$.

2.2 Some concepts from the theory of affine Lie algebras

Let us fix $k \in \mathbb{N}$ (the "level") and set $q := \exp(2\pi i/(k + c_{\mathfrak{g}})) \in U(1)$

- Set

$$\Lambda_+^k := \{\lambda \in \Lambda_+ \mid (\lambda, \theta) \leq k\} \tag{5}$$

- Let $\text{Isom}(\mathfrak{t})$ denote the group of isometries of the Euclidean vector space $(\mathfrak{t}, (\cdot, \cdot))$ and let $i : \text{Isom}(\mathfrak{t}) \to \text{Isom}(\mathfrak{t})$ denote the automorphism of $\text{Isom}(\mathfrak{t})$ given by

$$i(\tau)(b) = (k + c_{\mathfrak{g}}) \cdot \tau((b + \rho)/(k + c_{\mathfrak{g}})) - \rho \tag{6}$$

for all $b \in \mathfrak{t}$ and $\tau \in \text{Isom}(\mathfrak{t})$. We set[5]

$$\mathcal{W}_k := i(\mathcal{W}_{\text{aff}}) \subset \text{Isom}(\mathfrak{t}) \tag{7}$$

("(ρ-shifted) quantum Weyl group corresponding to the level k") and

$$\text{sgn}(\tau) := \text{sgn}(i^{-1}(\tau)) \quad \text{for } \tau \in \mathcal{W}_k$$

[3]Equivalently, one can define $\mathcal{W}_{\text{aff}}$ as the group of isometries of $\mathfrak{t} \cong \mathfrak{t}^*$ generated by $\mathcal{W}$ and the translations associated to the inverse roots

[4] note that $c_{\mathfrak{g}} = 1 + (\theta, \rho) = \frac{1}{2}(\theta, \theta + 2\rho) = \frac{1}{2} C_2(\theta)$. If we had normalized the Killing form $(\cdot, \cdot)$ such that the long roots have length 1 we would have $c_{\mathfrak{g}} = C_2(\theta)$, i.e. $c_{\mathfrak{g}}$ would then be the Casimir element associated to the adjoint representation.

[5]$\mathcal{W}_k$ coincides with the subgroup of $\text{Isom}(\mathfrak{t})$ which is generated by the orthogonal reflections on the ρ-shifted hyperplanes $H_\alpha - \rho$, $\alpha \in \mathcal{R}_+$, and the hyperplane $\{y \in \mathfrak{t} \mid (y, \theta) = k + c_{\mathfrak{g}}\} - \rho = \{x \in \mathfrak{t} \mid (x, \theta) = k + 1\}$, thus $\mathcal{W}_k$ is the same as the group $\mathcal{W}_0$ in [43].

- Let C, S, and T be the $\Lambda_+^k \times \Lambda_+^k$ matrices with complex entries given by

$$C_{\lambda\mu} := \delta_{\lambda\mu^*}, \tag{8}$$

$$T_{\lambda\mu} := \delta_{\lambda\mu}\, e^{\frac{\pi i C_2(\lambda)}{k+c_{\mathfrak{g}}}} \cdot e^{-\frac{\pi i c}{12}}, \tag{9}$$

$$S_{\lambda\mu} := \frac{i^{|\mathcal{R}_+|}}{(k+c_{\mathfrak{g}})^{r/2}} |\Lambda/\Lambda_{\check{\mathcal{R}}}|^{-\frac{1}{2}} \sum_{w\in\mathcal{W}} \operatorname{sgn}(w) e^{\frac{2\pi i}{k+c_{\mathfrak{g}}}(\lambda+\rho, w\cdot(\mu+\rho))} \tag{10}$$

for λ, μ where $c := \dim(\mathfrak{g}) \cdot \frac{k}{(k+c_{\mathfrak{g}})}$ (the "central charge").

One can prove by elementary methods that

$$S^2 = C \tag{11a}$$

$$(ST)^3 = C. \tag{11b}$$

In particular, S is invertible.

- For $\lambda \in \Lambda_+^k$ we set

$$\dim\lambda := \frac{S_{\lambda 0}}{S_{00}} \overset{(*)}{=} \prod_{\alpha\in\mathcal{R}_+} \frac{\sin\frac{\pi(\lambda+\rho,\alpha)}{k+c_{\mathfrak{g}}}}{\sin\frac{\pi(\rho,\alpha)}{k+c_{\mathfrak{g}}}} \tag{12}$$

Here $(*)$ follows from $\frac{S_{\lambda 0}}{S_{00}} = \frac{A(\lambda+\rho)(\rho)}{A(\rho)(\rho)}$ and the relation[6] $\delta(b) = A(\rho)(b)$ where we have set

$$A(\lambda)(b) := \sum_{w\in\mathcal{W}} \operatorname{sgn}(w) e^{2\pi i(\lambda, w\cdot b)}$$

$$\delta(b) := \prod_{\beta\in\mathcal{R}_+} (e^{\pi i\beta(b)} - e^{-\pi i\beta(b)}) = \prod_{\beta\in\mathcal{R}_+} (2i\sin(\pi(\beta,b))).$$

- For $\lambda, \mu, \nu \in \Lambda_+^k$ we define the "fusion coefficients" $N_{\lambda\mu\nu}$ and $N_{\mu\nu}^{\lambda}$ by

$$N_{\lambda\mu\nu} := \sum_{\sigma\in\Lambda_+^k} \frac{S_{\lambda\sigma}S_{\mu\sigma}S_{\nu\sigma}}{S_{0\sigma}} \tag{13}$$

and

$$N_{\mu\nu}^{\lambda} := N_{\lambda^*\mu\nu} \tag{14}$$

Observe that Eq. (11a) implies $N_{\mu 0}^{\nu} = \delta_{\mu}^{\nu}$.

Remark 2 Let us motivate the use of the term "fusion coefficients" above. Let $\hat{\mathfrak{g}}$ denote the affine Lie algebra corresponding to $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and let $\hat{N}_{\lambda\mu}^{\nu}$ be the fusion coefficients in the modular tensor categories based on the integrable representations of $\hat{\mathfrak{g}}$ at level k. Similarly, let $\check{N}_{\lambda\mu}^{\nu}$ be the fusion coefficients in the modular tensor category constructed in [5, 6] using the representation theory of the quantum group $U_q(\mathfrak{g}_{\mathbb{C}})$. Then we have

$$\hat{N}_{\mu\nu}^{\lambda} = N_{\mu\nu}^{\lambda} \tag{15}$$

$$\hat{N}_{\mu\nu}^{\lambda} = \check{N}_{\mu\nu}^{\lambda} \tag{16}$$

Eq. (15) are the famous "fusion rules" (cf., e.g., [27]) and Eq. (16) was proven in [24].

In [28, 46, 51] it was proven that $\hat{N}_{\gamma\alpha}^{\beta} = \sum_{\tau\in\mathcal{W}_k} \operatorname{sgn}(\tau) m_{\gamma}(\alpha - \tau(\beta))$. In view of Eq. (16) this is clearly equivalent to $\check{N}_{\gamma\alpha}^{\beta} = \sum_{\tau\in\mathcal{W}_k} \operatorname{sgn}(\tau) m_{\gamma}(\alpha - \tau(\beta))$. The latter formula[7] can be considered to be

[6] cf. [17] Theorem 1.7 in Chap. VI; cf. also the definition of the function $e(x)$ on p. 240 in [17]

[7] which was proved directly, i.e. without the use of Eq. (16), in [6, 44]

a quantum group analogue of the classical Racah formula. Following [44] we will call this formula the "(abstract) quantum Racah formula". The equivalent formula

$$N_{\gamma\alpha}^{\beta} = \sum_{\tau \in \mathcal{W}_k} \operatorname{sgn}(\tau) m_\gamma(\alpha - \tau(\beta)) \tag{17}$$

will be called[8] the "elementary quantum Racah formula".

3 The shadow invariant for links in $\Sigma \times S^1$

3.1 Definition

Let Σ be an oriented surface, let $L = (l_1, l_2, \dots, l_n)$, $n \in \mathbb{N}$, be a sufficiently regular link in $\Sigma \times S^1$, and let $l^j_{S^1}$ resp. l^j_Σ denote the projection of the loop l_j onto the S^1-component resp. Σ-component of the product $\Sigma \times S^1$. L can be turned into a framed link by picking for each loop l_j the standard framing described in Sec. 4 c) in [48] (this framing was called "vertical framing" in [32]). We also assume that each loop l_j is colored with an element γ_j of Λ_+^k.

We set $D(L) := (DP(L), E(L))$ where $DP(L)$ denotes the set of double points of L, i.e. the set of points $p \in \Sigma$ where the loops l^j_Σ, $j \leq n$, cross themselves or each other, and $E(L)$ the set of curves in Σ into which the loops $l^1_\Sigma, l^2_\Sigma, \dots, l^n_\Sigma$ are decomposed when being "cut" in the points of $DP(L)$. Clearly, $D(L)$ can be considered to be a finite (multi-)graph. We set $\Sigma \backslash D(L) := \Sigma \backslash (\bigcup_j \operatorname{arc}(l^j_\Sigma))$. We assume that the set of connected components of $\Sigma \backslash D(L)$ has only finitely many elements $Y_0, Y_1, Y_2, \dots, Y_\mu$, $\mu \in \mathbb{N}$, which we will call the "faces" of $\Sigma \backslash D(L)$.
As explained in [48] one can associate in a natural way a number $\operatorname{gl}(Y_t) \in \mathbb{Z}$, called "gleam" of Y_t, to each face Y_t (for an explicit formula for the gleams in the special cases that will be relevant for us later see Eq. (23) below). We call $X_L := (D(L), (\operatorname{gl}(Y_t))_{0 \leq t \leq \mu})$ the "shadow" of L.

Let $g \in E(L)$ be a fixed edge of the graph $D(L)$. Note that, as each loop l^j is oriented, g is an oriented curve in Σ. On the other hand, as Σ was assumed to be oriented, each face $Y \in \{Y_0, Y_1, Y_2, \dots, Y_\mu\}$ is an oriented surface and therefore also induces an orientation on its boundary ∂Y.

There is a unique face Y, denoted by Y_g^+ (resp. Y_g^-) in the sequel, such that $\operatorname{arc}(g) \subset \partial Y$ and, additionally, the orientation on $\operatorname{arc}(g)$ described above coincides with (resp. is opposite to) the orientation which is obtained by restricting the orientation on ∂Y to g. In other words: Y_g^+ and Y_g^- are the two[9] faces that "touch" the edge g, and Y_g^+ (resp. Y_g^-) is the face lying "to the left" (resp. "to the right") of g, cf. Fig. 1.

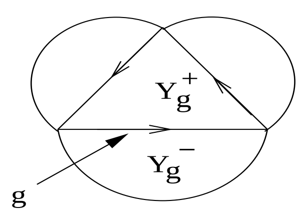

Figure 1:

A mapping $\varphi : \{Y_0, Y_1, Y_2, \dots, Y_\mu\} \to \Lambda_+^k$ will be called an area coloring of X_L (with colors in Λ_+^k) and the set of all such area colorings will be denoted by $col(X_L)$. We can now define the shadow invariant $|\cdot|$ by[10]

$$|X_L| := \sum_{\varphi \in \operatorname{col}(X_L)} |X_L|_1^\varphi \, |X_L|_2^\varphi \, |X_L|_3^\varphi \, |X_L|_4^\varphi \tag{18}$$

[8]Since for the derivation of (17) we used both the fusion rules (15) and the (abstract) quantum-Racah formula this name might be a little bit misleading. We could equally well call (17) the "elementary fusion rules"

[9] note that if Assumption 2 below is not fulfilled then possibly $Y_g^+ = Y_g^-$, so in this case there is actually only one such face

[10]this coincides with the definition in [47] up to an overall normalization factor which will be irrelevant for our purposes

where

$$|X_L|_1^\varphi = \prod_Y (\dim(\varphi(Y)))^{\chi(Y)} \tag{19}$$

$$|X_L|_2^\varphi = \prod_Y (v_{\varphi(Y)})^{\mathrm{gl}(Y)} \tag{20}$$

$$|X_L|_3^\varphi = \prod_{g \in E(L)} N^{\varphi(Y_g^-)}_{co(g)\varphi(Y_g^+)}. \tag{21}$$

where N^i_{jl} and $\dim(\cdot)$ are as in Subsec. 2.2, where $co(g)$ denotes the color associated to the edge g (i.e. $co(g) = \gamma_i$ where $i \leq n$ is given by $\mathrm{arc}(l^i_\Sigma) \supset g$) and where we have set $v_\lambda := T_{\lambda\lambda}$ (here T is, of course, the T-matrix from Subsec. 2.2). $|X_L|_4^\varphi$ is defined in terms of quantum 6j-symbols, cf. Chap. X, Sec. 1.2 in [47]. In view of the Assumption 1 below and the consequences that this assumption has, cf. Eq. (24) below, the precise definition of $|X_L|_4^\varphi$ in the general case will not be relevant in the present paper.

For the rest of this paper, we will restrict ourselves to the special situation where L also fulfills the following two assumptions.

Assumption 1 *The colored link L has no double points, i.e. the projected loops $l^1_\Sigma, l^2_\Sigma, \ldots, l^n_\Sigma$ are non-intersecting Jordan loops in Σ.*

Assumption 2 *Each l^j_Σ is 0-homologous.*

Assumptions 1 and 2 have the following consequences:

- For each $j \leq n$ the set $\Sigma \backslash \mathrm{arc}(l^j_\Sigma)$ has exactly two connected components. In the sequel R^+_j (resp. R^-_j) will denote the connected component "to the left" (resp. "to the right") of l^j_Σ, i.e. R^+_j (resp. R^-_j) is the unique connected component containing Y^+_j (resp. Y^-_j) where we have set

$$Y^\pm_j := Y^\pm_{l^j_\Sigma} \tag{22}$$

  (i.e. $Y^\pm_j = Y^\pm_g$ where $g = l^j_\Sigma$).

- $\mu = n$, i.e. $\Sigma \backslash (\bigcup_j \mathrm{arc}(l^j_\Sigma))$ has $n+1$ connected components $Y_0, Y_1, \ldots, Y_n$

- For each $Y \in \{Y_0, Y_1, Y_2, \ldots, Y_n\}$ we have

$$\mathrm{gl}(Y) = \sum_{j \text{ with } \mathrm{arc}(l^j_\Sigma) \subset \partial Y} \mathrm{wind}(l^j_{S^1}) \cdot \mathrm{sgn}(Y; l^j_\Sigma) \tag{23}$$

  where $\mathrm{wind}(l^j_{S^1})$ is the winding number of the loop $l^j_{S^1}$ and where $\mathrm{sgn}(Y; l^j_\Sigma)$ is given by

$$\mathrm{sgn}(Y; l^j_\Sigma) := \begin{cases} 1 & \text{if } Y \subset R^+_j \\ -1 & \text{if } Y \subset R^-_j \end{cases}$$

- According to the general definition of the shadow invariant in Chap. X, Sec. 1.2 in [47]. Assumption 1 implies $|X_L|_4^\varphi = 1$ so Eq. (18) reduces to

$$|X_L| = \sum_{\varphi \in \mathrm{col}(X_L)} |X_L|_1^\varphi |X_L|_2^\varphi |X_L|_3^\varphi \tag{24}$$

- "Vertical" framing for a loop l_j in $\Sigma \times S^1$ (cf. the first paragraph of the present subsection) is equivalent to what was called "horizontal" framing in Subsec. 5.2 in [32]

Remark 3 1. The "shadow invariant" defined in [47] is more general than what we have defined here above. Our definition is the special case of Turaev's shadow invariant where the underlying modular tensor category is the one coming from the representation theory of the quantum groups $U_q(\mathfrak{g}_{\mathbb{C}})$, cf. Remark 2 above.

2. In the special case $G = SU(2)$ one has $N_{jk}^i \in \{0,1\}$ for all $i,j,k \in \Lambda_+^k$ so $|X_L|_3^\varphi \in \{0,1\}$ for each $\varphi \in col(X_L)$. Let us call $\varphi \in col(X_L)$ "admissible" iff $|X_L|_3^\varphi = 1$ and set $col_{adm}(X_L) := \{\varphi \in col(X_L) \mid \varphi \text{ admissible } \}$. Then we can rewrite Eq. (18) in the form

$$|X_L| := \sum_{\varphi \in col_{adm}(X_L)} |X_L|_1^\varphi \, |X_L|_2^\varphi \, |X_L|_4^\varphi \tag{25}$$

If one compares this formula with Eqs. (5.7) and (5.8) in [48] (and the two equations before Theorem 6.1 in [48]) it is easy to see that the "shadow invariant" that was defined in [48] (and used in [32]) is the special case of the shadow invariant in the present paper which one obtains by taking $G = SU(2)$.

3.2 Some examples

Example 1 Let $\Sigma = S^2$ and let $L = ((l_1,l_2,l_3),(\lambda,\mu,\nu))$ be a colored link in $\Sigma \times S^1$ such that $\mathrm{wind}(l_{S^1}^i) = 1$ for all $i = 1,2,3$ and such that the projection of L onto the surface Σ looks like in the following figure. Let, for $i \in \{1,2,3\}$, Y_i denote the face "enclosed" by l_Σ^i and let Y_0 denote the remaining face. Clearly, we have $\chi(Y_i) = 1$ for $i \in \{1,2,3\}$ and $\chi(Y_0) = 2 - 2g - 3 = -1$ and $\mathrm{gl}(Y_i) = 1$ for $i \in \{1,2,3\}$ and $\mathrm{gl}(Y_0) = -3$. So we obtain

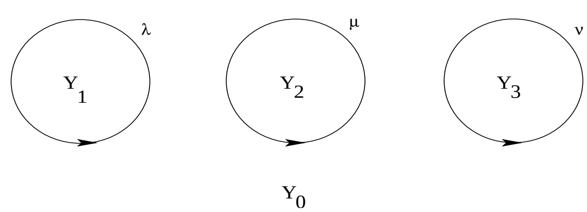

Figure 2:

$$\begin{aligned} |X_L| &= \sum_{\sigma_1\sigma_2\sigma_3\sigma_0 \in \Lambda_+^k} \mathrm{dim}(\sigma_1)\mathrm{dim}(\sigma_2)\mathrm{dim}(\sigma_3)(\mathrm{dim}(\sigma_0))^{-1} \, N_{\sigma_1\lambda}^{\sigma_0} N_{\sigma_2\mu}^{\sigma_0} N_{\sigma_3\nu}^{\sigma_0} \, T_{\sigma_1\sigma_1} T_{\sigma_2\sigma_2} T_{\sigma_3\sigma_3} T_{\sigma_0\sigma_0}^{-3} \\ &= \frac{T_{\lambda\lambda}T_{\mu\mu}T_{\nu\nu}}{T_{00}^3 S_{00}^2} \, N_{\lambda\mu\nu} \ . \end{aligned} \tag{26}$$

In deriving the last line, we used the following equation three times

$$\sum_{\lambda \in \Lambda_+^k} \mathrm{dim}(\lambda) \, T_{\lambda\lambda} \, N_{\mu\lambda}^\nu \quad = \quad \frac{1}{T_{00}S_{00}} \, (TST)_{\mu\nu} \tag{27}$$

(Eq. (27) follows from (11) and (13)).

Example 2 Let again $\Sigma = S^2$ and let $L = ((l_1,l_2,l_3),(\lambda,\mu,\nu))$ be a colored link in $\Sigma \times S^1$ such that $\mathrm{wind}(l_{S^1}^i) = 1$ for all $i = 1,2,3$ and such that the projection of L onto the surface Σ looks like in Fig. 3. Then we have $\chi(Y_1) = \chi(Y_3) = 1$, $\chi(Y_0) = \chi(Y_2) = 0$ and $\mathrm{gl}(Y_1) = \mathrm{gl}(Y_3) = 1$, $\mathrm{gl}(Y_0) = -2$, $\mathrm{gl}(Y_2) = 0$ where the faces Y_0, Y_1, Y_2, Y_3 are given as in Fig. 3. One obtains

$$|X_L| = \sum_{\sigma_1\sigma_2\sigma_3\sigma_0 \in \Lambda_+^k} \mathrm{dim}(\sigma_1) \, \mathrm{dim}(\sigma_3) \, N_{\nu\sigma_1}^{\sigma_2} N_{\lambda\sigma_2}^{\sigma_0} N_{\mu\sigma_3}^{\sigma_0} \, T_{\sigma_1\sigma_1} T_{\sigma_3\sigma_3} T_{\sigma_0\sigma_0}^{-2}. \tag{28}$$

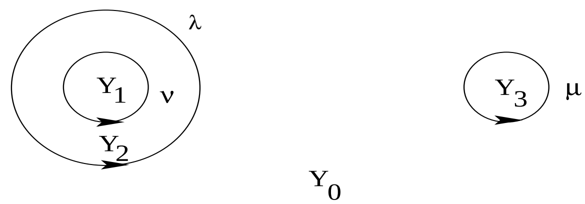

Figure 3:

The sums over σ_1 and σ_3 can be performed right away using twice Eq. (27). We get

$$|X_L| = \frac{T_{\mu\mu}T_{\nu\nu}}{T_{00}^2 S_{00}^2} \sum_{\sigma_2\sigma_0} T_{\sigma_2\sigma_2} T^{-1}_{\sigma_0\sigma_0} S_{\nu\sigma_2} S_{\mu\sigma_0} N^{\sigma_0}_{\lambda\sigma_2} \ . \tag{29}$$

Now observe that

$$\sum_{\sigma_2\sigma_0} T_{\sigma_2\sigma_2} T^{-1}_{\sigma_0\sigma_0} S_{\nu\sigma_2} S_{\mu\sigma_0} N^{\sigma_0}_{\lambda\sigma_2} \overset{(*)}{=} \frac{1}{T_{\nu\nu}} \sum_{\sigma_0\sigma} T^{-1}_{\sigma_0\sigma_0} T^{-1}_{\sigma\sigma} S_{\mu\sigma_0} S^{-1}_{\sigma_0\sigma} S_{\sigma\lambda} S_{\nu\sigma} \frac{1}{S_{\sigma 0}} \overset{(**)}{=} \frac{T_{\mu\mu}}{T_{\nu\nu}} N_{\lambda\mu\nu} \tag{30}$$

Here step $(*)$ follows from Eq. (13) and $STS = T^{-1}ST^{-1}$ (which in turn follows from Eq. (11)) and step $(**)$ follows from Eq. (13) and $ST^{-1}S^{-1} = TST$. From Eqs. (29) and (30) we finally get

$$|X_L| = \frac{T^2_{\mu\mu}}{T_{00}^2 S_{00}^2} N_{\lambda\mu\nu} \ . \tag{31}$$

Example 3 Note that X_L is also defined if L is the "empty" link $\emptyset$. In this case one has

$$|X_\emptyset| = \sum_{\lambda\in\Lambda^k_+} (\dim\lambda)^{2-2g} \ . \tag{32}$$

where g is the genus of the surface Σ.

4 State sums from the Chern-Simons path integral in the torus gauge

4.1 Chern-Simons models

Let M be an oriented compact 3-manifold and $\mathcal{A}$ the space of smooth $\mathfrak{g}$-valued 1-forms on M. Without loss of generality we can assume that the group G fixed in Subsec. 2.1 above is a Lie subgroup of $U(N)$, $N \in \mathbb{N}$. The Lie algebra $\mathfrak{g}$ of G can then be identified with the obvious Lie subalgebra of the Lie algebra $u(N)$ of $U(N)$.

The Chern-Simons action function S_{CS} associated to M, G, k (with k as in Subsec. 2.2) is given by

$$S_{CS}(A) = \tfrac{k}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A), \quad A \in \mathcal{A}$$

with $\mathrm{Tr} := c \cdot \mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}$ where the normalization constant c is chosen[11] such that

$$(A, B) = -\tfrac{1}{4\pi^2}\mathrm{Tr}(A \cdot B) \quad \forall A, B \in \mathfrak{g} \tag{33}$$

holds.[12] For example, if $G = SU(N)$ then $c = 1$ so in this case Tr coincides with $\mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}$.

[11] such a normalization is always possible because by assumption $\mathfrak{g}$ is simple so all Ad-invariant scalar products on $\mathfrak{g}$ are proportional to the Killing metric

[12] Here "$\cdot$" is, of course, the standard multiplication in $\mathrm{Mat}(N,\mathbb{C})$ and the wedge product $\wedge$ appearing in Eq. (33) is the one for $\mathrm{Mat}(N,\mathbb{C})$-valued forms.

From the definition of S_{CS} it is obvious that S_{CS} is invariant under (orientation-preserving) diffeomorphisms. Thus, at a heuristic level, we can expect that the heuristic integral (the "partition function") $Z(M) := \int \exp(iS_{CS}(A))DA$ is a topological invariant of the 3-manifold M. Here DA denotes the informal "Lebesgue measure" on the space $\mathcal{A}$.

Similarly, we can expect that the mapping which maps every sufficiently "regular" colored link $L = ((l_1, l_2, \ldots, l_n), (\gamma_1, \gamma_2, \ldots, \gamma_n))$ in M to the heuristic integral (the "Wilson loop observable" associated to L)

$$\mathrm{WLO}(L) := \frac{1}{Z(M)} \int \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} A\big)\big) \exp(iS_{CS}(A))DA \tag{34}$$

is a link invariant (or, rather, an invariant of colored links). Here we have set $\rho_i := \rho_{\gamma_i}$ $i \le n$, (cf. Subsec. 2.1), Tr_{ρ_i} is the trace in the representation ρ_i, and $\mathcal{P}\exp\big(\int_{l_i} A\big)$ denotes the holonomy of A around the loop l_i.

Let us now consider the special case $M = \Sigma \times S^1$ where Σ is a closed oriented surface. Due to the well-known "equivalence" of Witten's invariants and the Reshetikhin/Tureav invariants (cf., e.g., [52]) and the equivalence of the Reshetikhin/Tureav invariants with the shadow invariant (cf. Theorem 3.3 in Chap. X in [47]) one can conclude that in this situation $\mathrm{WLO}(L)$ should coincide with $|X_L|$ up to a multiplicative constant (independent of the link). The value of this constant can be determined by looking at the special case $L = \emptyset$, i.e. where L is the "empty" link. As $\mathrm{WLO}(\emptyset) = 1$ one can conclude that $\mathrm{WLO}(L) = \frac{1}{|X_\emptyset|} \cdot |X_L|$ should hold. One of the goals of this paper is to show this formula directly (for the special situation where the link L fulfills Assumptions 1 and 2 above) by applying a suitable gauge fixing procedure to the Chern-Simons path integral. This generalizes[13] the treatment in [32].

4.2 Torus gauge fixing applied to Chern-Simons models

During the rest of this paper we will set $M := \Sigma \times S^1$ where Σ is a closed oriented surface. Moreover, we will fix an arbitrary point $\sigma_0 \in \Sigma$ and an arbitrary[14] point $t_0 \in S^1$.

By $\mathcal{A}_\Sigma$ (resp. $\mathcal{A}_{\Sigma,\mathfrak{t}}$) we will denote the space of smooth $\mathfrak{g}$-valued (resp. $\mathfrak{t}$-valued) 1-forms on Σ. $\frac{\partial}{\partial t}$ will denote the vector field on S^1 which is induced by the curve $i_{S^1} : [0,1] \ni t \mapsto e^{2\pi i t} \in S^1 \subset \mathbb{C}$ and dt the 1-form on S^1 which is dual to $\frac{\partial}{\partial t}$. We can lift $\frac{\partial}{\partial t}$ and dt in the obvious way to a vector field resp. a 1-form on M, which will also be denoted by $\frac{\partial}{\partial t}$ resp. dt. Every $A \in \mathcal{A}$ can be written uniquely in the form $A = A^\perp + A_0 dt$ with $A^\perp \in \mathcal{A}^\perp$ and $A_0 \in C^\infty(M, \mathfrak{g})$ where $\mathcal{A}^\perp$ is defined by

$$\mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\tfrac{\partial}{\partial t}) = 0\} \tag{35}$$

We say that $A \in \mathcal{A}$ is in the "T-torus gauge" if $A \in \mathcal{A}^\perp \oplus \{B dt \mid B \in C^\infty(\Sigma, \mathfrak{t})\}$.

By computing the relevant Faddeev-Popov determinant one obtains for every gauge-invariant function $\chi : \mathcal{A} \to \mathbb{C}$ (cf. Eqs. (4.10a) and (4.10b) in [31])

$$\int_{\mathcal{A}} \chi(A) DA \;=\; const. \int_{C^\infty(\Sigma,\mathfrak{t})} \Big[\int_{\mathcal{A}^\perp} \chi(A^\perp + B dt) DA^\perp\Big] \triangle(B) \det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B))_{|\mathfrak{t}^\perp}\big) DB \tag{36}$$

where $\triangle(B) := \big|\det(\partial/\partial t + \mathrm{ad}(B))\big|$. Here $DA^\perp$ denotes the (informal) "Lebesgue measure" on $\mathcal{A}^\perp$ and DB the (informal) "Lebesgue measure" on $C^\infty(\Sigma, \mathfrak{t})$.

In the special case where $\chi(A) = \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} A\big)\big) \exp(iS_{CS}(A))$ we then get

$$\begin{aligned}\mathrm{WLO}(L) \sim \int_{C^\infty(\Sigma,\mathfrak{t})} \Big[\int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} A^\perp + B dt\big)\big) \exp(iS_{CS}(A^\perp + B dt)) DA^\perp\Big] \\ \times \triangle(B) \det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B))_{|\mathfrak{t}^\perp}\big) DB\end{aligned} \tag{37}$$

[13] in [32] only for the case where $G = SU(2)$ and where each γ_j was the highest weight of the fundamental representation the full path integral was evaluated explicitly

[14] in order to simplify the notation somewhat we will later restrict ourselves to the special case where $t_0 = i_{S^1}(0) = 1$

Here and in the sequel $\sim$ denotes equality up to a multiplicative constant independent of L. Now

$$S_{CS}(A^{\perp} + Bdt) = \tfrac{k}{4\pi} \int_M \big[\mathrm{Tr}(A^{\perp} \wedge dA^{\perp}) + 2\mathrm{Tr}(A^{\perp} \wedge Bdt \wedge A^{\perp}) + 2\mathrm{Tr}(A^{\perp} \wedge dB \wedge dt)\big]$$

so $S_{CS}(A^{\perp} + Bdt)$ is quadratic in $A^{\perp}$ for fixed B, which means that the informal (complex) measure $\exp(iS_{CS}(A^{\perp} + Bdt))DA^{\perp}$ appearing above is of "Gaussian type". This increases the chances of making rigorous sense of the right-hand side of Eq. (37) considerably.

So far we have ignored the following two "subtleties"

1. When one tries to find a rigorous meaning for the informal measure resp. the corresponding integral functional in Eq. (37) above one encounters certain problems which can be solved by introducing a suitable decomposition $\mathcal{A}^{\perp} = \hat{\mathcal{A}}^{\perp} \oplus \mathcal{A}^{\perp}_c$, which we will describe now (for a detailed motivation of this decomposition, see Sec. 8 in [31] and Sec. 3.4 in [32]):

   Let us make the identification $\mathcal{A}^{\perp} \cong C^{\infty}(S^1, \mathcal{A}_{\Sigma})$ where $C^{\infty}(S^1, \mathcal{A}_{\Sigma})$ denotes the space of all "smooth" functions $\alpha : S^1 \to \mathcal{A}_{\Sigma}$, i.e. all functions $\alpha : S^1 \to \mathcal{A}_{\Sigma}$ with the property that every smooth vector field X on Σ the function $\Sigma \times S^1 \ni (\sigma, t) \mapsto \alpha(t)(X_{\sigma})$ is smooth.

   The decomposition $\mathcal{A}^{\perp} = \hat{\mathcal{A}}^{\perp} \oplus \mathcal{A}^{\perp}_c$ is defined by[15]

   $$\hat{\mathcal{A}}^{\perp} := \{A^{\perp} \in C^{\infty}(S^1, \mathcal{A}_{\Sigma}) \mid \pi_{\mathcal{A}_{\Sigma,\mathfrak{t}}}(A^{\perp}(t_0)) = 0\}, \tag{38}$$
   $$\mathcal{A}^{\perp}_c := \{A^{\perp} \in C^{\infty}(S^1, \mathcal{A}_{\Sigma}) \mid A^{\perp} \text{ is constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}\text{-valued}\} \tag{39}$$

   where $\pi_{\mathcal{A}_{\Sigma,\mathfrak{t}}} : \mathcal{A}_{\Sigma} \to \mathcal{A}_{\Sigma,\mathfrak{t}}$ is the projection onto the first component in the decomposition $\mathcal{A}_{\Sigma} = \mathcal{A}_{\Sigma,\mathfrak{t}} \oplus \mathcal{A}_{\Sigma,\mathfrak{t}^{\perp}}$. It turns out that S_{CS} behaves nicely under this decomposition. More precisely, we have

   $$S_{CS}(\hat{A}^{\perp} + A^{\perp}_c + Bdt) = S_{CS}(\hat{A}^{\perp} + Bdt) + \frac{k}{2\pi} \int_{\Sigma} \mathrm{Tr}(dA^{\perp}_c \cdot B) \tag{40}$$

   Using this and setting $d\hat{\mu}^{\perp}_B(\hat{A}^{\perp}) := \frac{1}{\hat{Z}(B)} \exp(iS_{CS}(\hat{A}^{\perp}+Bdt))D\hat{A}^{\perp}$ where $\hat{Z}(B) := \int \exp(iS_{CS}(\hat{A}^{\perp}+Bdt))D\hat{A}^{\perp}$ we obtain

   $$\begin{aligned} \mathrm{WLO}(L) \sim & \\ & \int_{C^{\infty}(\Sigma,\mathfrak{t})} \int_{\mathcal{A}^{\perp}_c} \Big[\int_{\hat{\mathcal{A}}^{\perp}} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} (\hat{A}^{\perp} + A^{\perp}_c + Bdt)\big)\big) d\hat{\mu}^{\perp}_B(\hat{A}^{\perp})\Big] \\ & \quad \times \exp(i\tfrac{k}{2\pi} \int_{\Sigma} \mathrm{Tr}(dA^{\perp}_c \cdot B)) DA^{\perp}_c \det\big(1_{\mathfrak{t}^{\perp}} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^{\perp}})\big) \triangle[B] \hat{Z}(B)\big\} DB \end{aligned} \tag{41}$$

   A more careful analysis shows that in the formula above one can replace $\mathfrak{t}$ by $\mathfrak{t}_{reg}$ or, alternatively, by the Weyl alcove P. This amounts to including the extra factor $1_{C^{\infty}(\Sigma,\mathfrak{t}_{reg})}$ or $1_{C^{\infty}(\Sigma,P)}$ in the integral expression above. In the sequel we will use the factor $1_{C^{\infty}(\Sigma,P)}$.

2. If one studies the torus gauge fixing procedure more closely one finds that – due to certain topological obstructions (cf. [14], [31]) – in general a 1-form A can be gauge-transformed into a 1-form of the type $A^{\perp} + Bdt$ only if one uses a gauge transformation Ω which has a certain (mild) singularity and if one allows $A^{\perp}$ to have a similar singularity. Concretely, in [32] we worked with gauge transformations Ω of the type $\Omega = \Omega_{smooth} \cdot \Omega_{\mathrm{sing}}(\mathrm{h}) \in C^{\infty}((\Sigma \backslash \{\sigma_0\}) \times S^1, G)$ with $\Omega_{smooth} \in C^{\infty}(\Sigma \times S^1, G)$ and $\Omega_{\mathrm{sing}}(\mathrm{h}) \in C^{\infty}(\Sigma \backslash \{\sigma_0\}, G) \subset C^{\infty}((\Sigma \backslash \{\sigma_0\}) \times S^1, G)$ where $\sigma_0 \in \Sigma$ is the point fixed above and where the parameter h is an element of $[\Sigma, G/T]$, i.e. a homotopy class of mappings from Σ to G/T. $\Omega_{\mathrm{sing}}(\mathrm{h})$ is obtained from h by fixing a representative $\bar{g}(\mathrm{h}) \in C^{\infty}(\Sigma, G/T)$ of h and then lifting the restriction $\bar{g}(\mathrm{h})_{|\Sigma \backslash \{\sigma_0\}} : \Sigma \backslash \{\sigma_0\} \to G/T$ to a mapping $\Sigma \backslash \{\sigma_0\} \to G$. In other words:

[15] note that the space $\hat{\mathcal{A}}^{\perp}$ depends on the choice of the point t_0, and so will some expressions appearing later, see e.g. Eq. (47) below

$\Omega_{\text{sing}}(\mathrm{h}) \in C^\infty(\Sigma\backslash\{\sigma_0\}, G)$ is a fixed mapping with the property that $\pi_{G/T} \circ \Omega_{\text{sing}}(\mathrm{h}) = \bar{g}(\mathrm{h})_{|\Sigma\backslash\{\sigma_0\}}$ where $\pi_{G/T} : G \to G/T$ is the canonical projection.

The use of the singular gauge transformations $\Omega_{\text{sing}}(\mathrm{h})$ gives rise to an extra summation $\sum_{\mathrm{h}\in[\Sigma,G/T]}$ and to extra terms $A^\perp_{sing}(\mathrm{h}) := \pi_{\mathfrak{t}}(\Omega_{\text{sing}}(\mathrm{h})^{-1} \cdot d\Omega_{\text{sing}}(\mathrm{h}))$, i.e. in Eq. (41) above we have to include a summation $\sum_{\mathrm{h}\in[\Sigma,G/T]}$ and we have to replace $A^\perp_c$ by $A^\perp_c + A^\perp_{sing}(\mathrm{h})$ (for a detailed description and justification of all this, see [32]).

Taking into account these two subtleties we obtain

$$\begin{aligned} &\mathrm{WLO}(L) \sim \\ &\sum_{\mathrm{h}\in[\Sigma,G/T]} \int_{\mathcal{A}^\perp_c \times C^\infty(\Sigma,\mathfrak{t})} 1_{C^\infty(\Sigma,P)}(B) \Big[\int_{\hat{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} (\hat{A}^\perp + A^\perp_c + A^\perp_{sing}(\mathrm{h}) + Bdt)\big)\big) d\hat{\mu}^\perp_B(\hat{A}^\perp)\Big] \\ &\qquad \times \big\{\exp(i\tfrac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_{sing}(\mathrm{h}) \cdot B)) \det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\triangle[B]\hat{Z}(B)\big\} \\ &\qquad\qquad \times \exp(i\tfrac{k}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c \cdot B))(DA^\perp_c \otimes DB) \end{aligned} \tag{42}$$

where

$$\int_\Sigma \mathrm{Tr}(dA^\perp_{sing}(\mathrm{h}) \cdot B) := \lim_{\epsilon\to 0} \int_{\Sigma\backslash B_\epsilon(\sigma_0)} \mathrm{Tr}(dA^\perp_{sing}(\mathrm{h}) \cdot B)$$

Here $B_\epsilon(\sigma_0)$ is the closed ϵ-ball around σ_0 with respect to an arbitrary but fixed Riemannian metric on Σ.

Remark 4 The mapping $n : [\Sigma, G/T] \to \mathfrak{t}$ given by $n(\mathrm{h}) = \lim_{\epsilon\to 0}\int_{\Sigma\backslash B_\epsilon(\sigma_0)} A^\perp_{sing}(\mathrm{h})$ is independent of the special choice of $\bar{g}(\mathrm{h})$ and $\Omega_{sing}(\mathrm{h})$, cf. [14, 33]. Moreover, and this will be important in Subsec. 4.4 below one can show that n is a bijection from $[\Sigma, G/T]$ onto $I = \ker(\exp_{|\mathfrak{t}})$.

4.3 Some comments regarding a rigorous realization of the r.h.s. of Eq. (42)

In [32] it is explained how one can make rigorous sense of the path integral expression appearing on the right-hand side of Eq. (42) using results/constructions from White noise analysis (cf., e.g., [36]), certain regularization techniques like "loop smearing" and "framing", and a suitable regularization of the expression $\det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\triangle[B]\hat{Z}(B)$ appearing above. We do not want to repeat the discussion in [32] in the present paper. Let us just remark the following:

1. In view of the results [12] it is clear how to make sense of the factor $\det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\triangle[B]\hat{Z}(B)$ appearing Eq. (42) in the special case where B is a constant function (this was the only case which was relevant in [12]). More precisely, using the same ζ-function regularization as the one described in Sec. 6 in [12] one comes to the conclusion that in this special case of constant $B \equiv b$ the expression $\det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\triangle[B]\hat{Z}(B)$ should be replaced by

$$\exp(i\tfrac{c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c \cdot B)) \times \det_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) \tag{43}$$

   where

$$\det_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) := \det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(b))_{|\mathfrak{t}^\perp}\big)^{\chi(\Sigma)/2}$$

   In [32] it was suggested that in the more general situation where B is a step function of the form $B = \sum_{t=0}^\mu b_t 1_{Y_t}$ one should again replace $\det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\triangle[B]\hat{Z}(B)$ by expression (43) where now

$$\det_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) := \prod_{t=0}^{\mu} \det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(b_t))_{|\mathfrak{t}^\perp}\big)^{\chi(Y_t)/2} \tag{44}$$

Moreover, it was suggested that one should include a $\exp(i\frac{c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_{sing}(\mathrm{h})\cdot B))$-factor in the expression (43) above. Incorporating these changes into (42) one obtains

$$\mathrm{WLO}(L) \sim$$
$$\sum_{\mathrm{h}\in[\Sigma,G/T]} \int_{\mathcal{A}^\perp_c\times C^\infty(\Sigma,\mathfrak{t})} 1_{C^\infty(\Sigma,P)}(B)\Big[\int_{\hat{\mathcal{A}}^\perp}\prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i}(\hat{A}^\perp + A^\perp_c + A^\perp_{sing}(\mathrm{h}) + Bdt)\big)\big)d\hat{\mu}^\perp_B(\hat{A}^\perp)\Big]$$
$$\times \big\{\exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_{sing}(\mathrm{h})\cdot B))\,\mathrm{det}_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\big\}d\nu((A^\perp_c, B)) \quad (45)$$

where we have introduced the heuristic complex measure $d\nu$ given by

$$d\nu((A^\perp_c, B)) := \exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c\cdot B))(DA^\perp_c\otimes DB)$$

2. The integral functionals $\int_{\hat{\mathcal{A}}^\perp}\cdots d\hat{\mu}^\perp_B(\hat{A}^\perp)$ resp. $\int_{\mathcal{A}^\perp_c\times C^\infty(\Sigma,\mathfrak{t})}\cdots d\nu((A^\perp_c, B))$ appearing above can be realized rigorously as Hida distributions $\Phi^\perp_B$ resp. Ψ on suitable extensions of the spaces $\hat{\mathcal{A}}^\perp$ and $\mathcal{A}^\perp_c\times C^\infty(\Sigma,\mathfrak{t})$. Moreover, also the space $C^\infty(\Sigma,P)$ appearing in the indicator function $1_{C^\infty(\Sigma,P)}$ must be replaced by a larger space. (The fact that one has to extend the original spaces of smooth functions by larger spaces consisting of less regular functions is a usual phenomenon in Constructive Quantum Field Theory.) The details regarding the extensions of the spaces $\hat{\mathcal{A}}^\perp$ and $\mathcal{A}^\perp_c\times C^\infty(\Sigma,\mathfrak{t})$ have been or will be discussed elsewhere[16] and they are not relevant if one is only interested in a heuristic evaluation of the r.h.s of Eq. (42) resp. (45). By contrast, the question of how to extend the space $C^\infty(\Sigma,P)$ appearing in the indicator function $1_{C^\infty(\Sigma,P)}$ is more subtle even if one is only interested in a heuristic treatment. One might think that if one replaces $C^\infty(\Sigma,P)$ by the space P^Σ of *all* P-valued functions on Σ this should be enough. In fact that was the ansatz used in [32] and in the special case where all the link colors γ_i are (minimal) fundamental weights this ansatz works. However, it turns out that in the case of general link colors γ_i the space P^Σ is too small. In order to find the "correct" space note that $1_{C^\infty(\Sigma,P)}(B) = 1_{C^\infty(\Sigma,\mathfrak{t}_{reg})}(B)1_P(B(\sigma_0))$. This suggests that one might try to replace $C^\infty(\Sigma,\mathfrak{t}_{reg})$ by $(\mathfrak{t}_{reg})^\Sigma$. As the computations in the next subsections show the second ansatz is the "correct" one. Of course, it would be desirable to find a thorough justification for using the second ansatz which is independent of the results in the rest of this paper.

3. For the implementation of the "framing procedure" in [32] a suitable family $(\phi_s)_{s>0}$ of diffeomorphisms of $\Sigma\times S^1$ fulfilling certain condition (see list below) was fixed. For each diffeomorphism ϕ_s a "deformation" $\Phi^\perp_{B,\phi_s}$ resp. $\Psi_{\bar{\phi}_s}$ of $\Phi^\perp_B$ resp. Ψ was then introduced and used to replace $\Phi^\perp_B$ and Ψ in the original formula. Later the free parameter $s>0$ in the resulting formulas was eliminated by taking the limit $s\to 0$.

   Among others $(\phi_s)_{s>0}$ was assumed to fulfill the following conditions

   - $\phi_s \to \mathrm{id}_M$ as $s\to 0$ uniformly w.r.t. to an arbitrary Riemannian metric on M.
   - $(\phi_s)^*\mathcal{A}^\perp = \mathcal{A}^\perp$ for all $s>0$. This condition implies that each ϕ_s, $s>0$, is of the form

     $$\phi_s(\sigma,t) = (\bar{\phi}_s(\sigma), v_s(\sigma,t)) \quad \forall(\sigma,t)\in\Sigma\times S^1$$

     for a uniquely determined diffeomorphism $\bar{\phi}_s:\Sigma\to\Sigma$ and $v_s\in C^\infty(\Sigma\times S^1,S^1)$.
   - $(\phi_s)_{s>0}$ is "horizontal" in the sense that[17] it can be obtained by integrating a smooth vector field X on $\Sigma\times S^1$, which for all $i\le n$, $u\in[0,1]$ is orthogonal to the tangent vector $l'_i(u)$ (i.e. $X(l_i(u))\perp l'_i(u)$) and, at the same time, horizontal in $l_i(u)$ (i.e. $dt(X(l_i(u)))=0$).

[16] the extension of $\hat{\mathcal{A}}^\perp$ is described in Sec. 8 in [31], see also Sec. 4 in [32]; the extension of $\mathcal{A}^\perp_c\times C^\infty(\Sigma,\mathfrak{t})$ in the special case where G is Abelian was described in the last remark in Subsec. 6.1 in [32]

[17] in fact the definition of the term "horizontal" in [32] was somewhat broader but also more complicated

4.4 Explicit heuristic evaluation of the WLOs

As mentioned above we will not go into details concerning a rigorous realization of the r.h.s. of (45) but give a short heuristic treatment instead. As the starting point for this heuristic treatment we use the following modification[18] of Eq. (45) above.

$$\begin{aligned}
&\mathrm{WLO}(L) \sim \\
&\sum_{\mathrm{h}\in[\Sigma,G/T]} \int 1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \Big[\int_{\mathcal{A}_c^\perp} \int_{\hat{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} (\hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + B dt)\big)\big) d\hat{\mu}_B^\perp(\hat{A}^\perp) \\
&\qquad \times \big\{\exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}(dA_{sing}^\perp(\mathrm{h}) \cdot B)) \det{}_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\big\} \\
&\qquad\qquad \times \exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp \Big] DB \qquad (46)
\end{aligned}$$

Let, for fixed $j \leq n$, $u_1, u_2, \ldots u_{n^j}$ be the "solutions" of the equation $l^j_{S^1}(u) = t_0$, i.e. those curve parameters in which $l^j_{S^1}$ "hits" t_0. For $m \leq n^j$ we set $\sigma^j_m := l^j_\Sigma(u_m)$, and $\epsilon^j_m := 1$ resp. $\epsilon^j_m := -1$ resp. $\epsilon^j_m := 0$ if $l^j_{S^1}$ crosses t_0 in u_m "from below" resp. "from above" resp. only touches t_0 in u_m.

In [32] it is shown how one can evaluate (the rigorous realization of) the heuristic expression

$$\int_{\hat{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} (\hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + B dt)\big)\big) d\hat{\mu}_B^\perp(\hat{A}^\perp)$$

explicitly (for fixed smooth B and $A_c^\perp$). One then obtains

$$\prod_{j=1}^n \mathrm{Tr}_{\rho_j}\Big[\exp(\int_{l^j_\Sigma} A_c^\perp) \exp(\int_{l^j_\Sigma} A_{sing}^\perp(\mathrm{h})) \exp(\textstyle\sum_{m=1}^{n^j} \epsilon^j_m B(\sigma^j_m))\Big]$$

Plugging this into Eq. (46) we obtain

$$\begin{aligned}
&\mathrm{WLO}(L) \sim \\
&\sum_{\mathrm{h}} \int 1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \Big[\int_{\mathcal{A}_c^\perp} \prod_{j=1}^n \mathrm{Tr}_{\rho_j}\Big[\exp(\int_{l^j_\Sigma} A_c^\perp) \exp(\int_{l^j_\Sigma} A_{sing}^\perp(\mathrm{h})) \exp(\textstyle\sum_{m=1}^{n^j} \epsilon^j_m B(\sigma^j_m))\Big] \\
&\qquad \times \big\{\exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}(dA_{sing}^\perp(\mathrm{h}) \cdot B)) \det{}_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\big\} \\
&\qquad\qquad \times \exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp \Big] DB \qquad (47)
\end{aligned}$$

Let us now fix an auxiliary Riemannian metric $\mathbf{g}$ on Σ for the rest of this paper. Let $\mu_{\mathbf{g}}$ denote the corresponding volume measure on Σ and $\star$ the Hodge star operator induced by $\mathbf{g}$. Moreover, let $L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})$ denote obvious[19] L^2-space. Then we have (cf. Eq. (33))

$$\int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B) = \int \mathrm{Tr}(\star dA_c^\perp \cdot B) d\mu_{\mathbf{g}} = -4\pi^2 \ll \star dA_c^\perp, B \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})}$$

From Stokes' Theorem we obtain

$$\int_{l^j_\Sigma} A_c^\perp = \int_{R_j^+} dA_c^\perp = \int_{R_j^+} \star dA_c^\perp d\mu_{\mathbf{g}} = \int \star dA_c^\perp \cdot 1_{R_j^+} d\mu_{\mathbf{g}}$$

[18] clearly, the modification consists in replacing the integration $\int \cdots d\nu$ by the two separate integrations $\int \cdots DB$ and $\int_{\mathcal{A}_c^\perp} \cdots DA_c^\perp$ and in the use of $1_{(\mathfrak{t}_{reg})^\Sigma}(B)\ 1_P(B(\sigma_0))$ instead of $1_{C^\infty(\Sigma,P)}(B) = 1_{C^\infty(\Sigma,\mathfrak{t}_{reg})}(B) 1_P(B(\sigma_0))$

[19] the inner product $\ll \cdot, \cdot \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})}$ is given by $\ll B_1, B_2 \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})} = \int_\Sigma (B_1(\sigma), B_2(\sigma)) d\mu_{\mathbf{g}}(\sigma)$ where $(\cdot,\cdot)$ is the inner product on $\mathfrak{g} \supset \mathfrak{t}$ fixed above.

which implies

$$(\alpha, \int_{l_\Sigma^j} A_c^\perp) = \ll \star dA_c^\perp, \alpha \cdot 1_{R_j^+} \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})} \tag{48}$$

for every $\alpha \in \mathfrak{t}$. Here $1_{R_j^+}$ denotes the indicator function of the region R_j^+ defined in Subsec. 3.1 above. Note that Eq. (48) also holds if we replace $1_{R_j^+}$ by

$$1_{R_j^+}^{\text{shift}} := 1_{R_j^+} - 1_{R_j^+}(\sigma_0) \tag{49}$$

We will use this modified version of Eq. (48) in the sequel. Setting

$$\bar{\alpha} := 2\pi\alpha, \tag{50}$$

for each $\alpha \in \Lambda$ we then obtain from Eq. (2) and the modified version of Eq. (48)

$$\begin{aligned}
&\mathrm{Tr}_{\rho_j}\Big[\exp(\int_{l_\Sigma^j} A_c^\perp)\exp(\int_{l_\Sigma^j} A_{sing}^\perp(\mathrm{h}))\exp(\sum\nolimits_m \epsilon_m^j B(\sigma_m^j))\Big] \\
&= \sum_{\alpha\in\Lambda} m_{\gamma_j}(\alpha)\exp(i(\bar{\alpha}, \sum\nolimits_m \epsilon_m^j B(\sigma_m^j)))) \cdot \exp(\int_{l_\Sigma^j} (\bar{\alpha}, A_{sing}^\perp(\mathrm{h}))) \cdot \exp(i \ll \star dA_c^\perp, \bar{\alpha}\cdot 1_{R_j^+}^{\text{shift}} \gg_{L^2_{\mathfrak{t}}(\Sigma,d\mu_{\mathbf{g}})})
\end{aligned}$$

(Here $(\bar{\alpha}, A_{sing}^\perp(\mathrm{h}))$ denotes the obvious real-valued 1-form). Plugging this into Eq. (47) above we get

$$\begin{aligned}
&\mathrm{WLO}(L) \\
&\sim \sum_{\mathrm{h}} \int 1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \int_{\mathcal{A}_c^\perp} \prod_{j=1}^n \Big[\sum_{\alpha_j\in\Lambda} m_{\gamma_j}(\alpha_j)\exp(\int_{l_\Sigma^j}(\bar{\alpha}_j, A_{sing}^\perp(\mathrm{h})))\exp(i\ll \star dA_c^\perp, \bar{\alpha}_j \cdot 1_{R_j^+}^{\text{shift}} \gg_{L^2_{\mathfrak{t}}(\Sigma,d\mu_{\mathbf{g}})}) \\
&\qquad \times \exp(i(\bar{\alpha}_j, \sum\nolimits_m \epsilon_m^j B(\sigma_m^j))))\Big] \det{}_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA_{sing}^\perp(\mathrm{h})\cdot B)) \\
&\qquad \times \exp(-2\pi i(k+c_{\mathfrak{g}}) \ll \star dA_c^\perp, B \gg_{L^2_{\mathfrak{t}}(\Sigma,d\mu_{\mathbf{g}})}) DA_c^\perp DB \\
&= \sum_{\mathrm{h}} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in\Lambda} (\prod_{j=1}^n m_{\gamma_j}(\alpha_j)) \int 1_{(\mathfrak{t}_{reg})^\Sigma}(B)1_P(B(\sigma_0)) \prod_{j=1}^n \exp(\int_{l_\Sigma^j}(\bar{\alpha}_j, A_{sing}^\perp(\mathrm{h})))\exp(i(\bar{\alpha}_j, \sum\nolimits_m \epsilon_m^j B(\sigma_m^j)))) \\
&\qquad \times \det{}_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)\exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA_{sing}^\perp(\mathrm{h})\cdot B)) \\
&\qquad \times \Big[\int_{\mathcal{A}_c^\perp} \exp(i \ll \star dA_c^\perp, \sum_{j=1}^n \bar{\alpha}_j \cdot 1_{R_j^+}^{\text{shift}} - 2\pi(k+c_{\mathfrak{g}})B \gg_{L^2_{\mathfrak{t}}(\Sigma,d\mu_{\mathbf{g}})}) DA_c^\perp\Big] DB \\
&\overset{(*)}{=} \sum_{\mathrm{h}} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in\Lambda} (\prod_{j=1}^n m_{\gamma_j}(\alpha_j)) \int 1_{(\mathfrak{t}_{reg})^\Sigma}(B)1_P(B(\sigma_0))\exp(i\tfrac{k+c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA_{sing}^\perp(\mathrm{h})\cdot B)) \\
&\qquad \times \det{}_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) \prod_{j=1}^n \Big(\exp(\int_{l_\Sigma^j}(\bar{\alpha}_j, A_{sing}^\perp(\mathrm{h})))\exp(i(\bar{\alpha}_j, \sum\nolimits_m \epsilon_m^j B(\sigma_m^j))))\Big) \\
&\qquad \times \Big[\delta\big(d(\sum_{j=1}^n \bar{\alpha}_j \cdot 1_{R_j^+}^{\text{shift}} - 2\pi(k+c_{\mathfrak{g}})B)\big)\Big] DB
\end{aligned} \tag{51}$$

Here, in step $(*)$ we have used the informal equation

$$\begin{aligned}
&\int_{\mathcal{A}_c^\perp} \exp(i \ll \star dA_c^\perp, \sum_{j=1}^n \bar{\alpha}_j \cdot 1_{R_j^+}^{\text{shift}} - 2\pi(k+c_{\mathfrak{g}})B \gg_{L^2_{\mathfrak{t}}(\Sigma,d\mu_{\mathbf{g}})}) DA_c^\perp \\
&\qquad\qquad \sim \delta\big(d(\sum_{j=1}^n \bar{\alpha}_j \cdot 1_{R_j^+}^{\text{shift}} - 2\pi(k+c_{\mathfrak{g}})B)\big)
\end{aligned} \tag{52}$$

which is a kind of infinite dimensional analogue of the well-known informal equation $\int_{\mathbb{R}} \exp(i\langle x, y\rangle)dx \sim \delta(y)$. In fact, as $\big(\sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+} - 2\pi(k+c_{\mathfrak{g}})B\big)$ is in general not smooth (not even continuous) we should be a little bit more careful. Instead of using the delta-function $\delta\big(d\big(\sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+} - 2\pi(k + c_{\mathfrak{g}})B\big)\big)$ in Eqs. (51) and (52) above we should rather use the "superposition"[20]

$$\int_{\mathfrak{t}} \cdots \delta\big(B - \big(b + \tfrac{1}{2\pi(k+c_{\mathfrak{g}})} \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big)\big)\big) db$$

of delta-functions. Then we obtain

$$\begin{aligned}
&\mathrm{WLO}(L) \\
&\sim \sum_{\mathrm{h} \in [\Sigma, G/T]} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda} \big(\prod_{j=1}^n m_{\gamma_j}(\alpha_j)\big) \int_{\mathfrak{t}} db \Big[\exp\big(i \tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}(dA^{\perp}_{sing}(\mathrm{h}) \cdot b)\big) \\
&\qquad \times \Big(1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \det\nolimits_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) \prod_{j=1}^n \exp\big(i(\bar{\alpha}_j, \textstyle\sum_m \epsilon^j_m B(\sigma^j_m))\big)\Big)_{|B = b + \frac{1}{2\pi(k+c_{\mathfrak{g}})} \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}} \\
&\qquad \times \Big\{ \exp\big(\sum_j \int_{l^j_\Sigma} (\bar{\alpha}_j, A^{\perp}_{sing}(\mathrm{h}))\big) \exp\big(i \tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}\big(dA^{\perp}_{sing}(\mathrm{h}) \cdot \tfrac{1}{2\pi(k+c_{\mathfrak{g}})} \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big) \Big\} \Big] \\
&\overset{(**)}{=} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda} \big(\prod_{j=1}^n m_{\gamma_j}(\alpha_j)\big) \sum_{\mathrm{h} \in [\Sigma, G/T]} \int_{\mathfrak{t}} db \Big[\exp\big(i 2\pi (k + c_{\mathfrak{g}})(n(\mathrm{h}), b)\big) \\
&\qquad \times \Big(1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \det\nolimits_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) \prod_{j=1}^n \exp\big(i(\bar{\alpha}_j, \textstyle\sum_m \epsilon^j_m B(\sigma^j_m))\big)\Big)_{|B = b + \frac{1}{2\pi(k+c_{\mathfrak{g}})} \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}} \\
&\qquad \times \Big\{ 1 \Big\} \Big] \\
&\overset{(***)}{=} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda} \big(\prod_{j=1}^n m_{\gamma_j}(\alpha_j)\big) \textstyle\sum_{b \in \frac{1}{k+c_{\mathfrak{g}}}\Lambda} \Big(1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \\
&\qquad \times \det\nolimits_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) \prod_{j=1}^n \exp\big(2\pi i(\alpha_j, \textstyle\sum_m \epsilon^j_m B(\sigma^j_m))\big)\Big)_{|B = b + \frac{1}{k+c_{\mathfrak{g}}} \sum_{j=1}^n \alpha_j \cdot 1^{\text{shift}}_{R_j^+}}
\end{aligned}$$

In step $(**)$ we have used the definition of $n(\mathrm{h})$ and Eq. (33). Moreover, we have used

$$\exp\big(i \tfrac{k+c_{\mathfrak{g}}}{2\pi} \int_\Sigma \mathrm{Tr}\big(dA^{\perp}_{sing}(\mathrm{h}) \cdot \tfrac{1}{2\pi(k+c_{\mathfrak{g}})} \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big)\big) = \exp\big(- \sum_j \int_{l^j_\Sigma} (\bar{\alpha}_j, A^{\perp}_{sing}(\mathrm{h}))\big)$$

(the last equation generalizes Eq. (6.39) in [32]). Step $(***)$ follows, informally by interchanging $\sum_{\mathrm{h}} \cdots$ and $\int_{\mathfrak{t}} db \cdots$ and then using

$$\sum_{x \in I} \exp(2\pi i(k + c_{\mathfrak{g}})(x, b)) = \sum_{b' \in \frac{1}{k+c_{\mathfrak{g}}} I^*} \delta(b - b')$$

which is an informal version of the Poisson summation formula (moreover, one has to take into account Remark 4 and the relation $I^* = \Lambda$, cf. Remark 1).

[20]one can equally well use the superposition $\int_{\mathfrak{t}} \cdots \delta\big(B - \frac{1}{2\pi(k+c_{\mathfrak{g}})}\big(-b + \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big)\big)\big) db$ or $\int_{\mathfrak{t}} \cdots \delta\big(B - \frac{1}{2\pi(k+c_{\mathfrak{g}})}\big(b + \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big)\big)\big) db$ the final result will be the same, which is not surprising since, heuristically, $\delta\big(d\big(\sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+} - 2\pi(k + c_{\mathfrak{g}})B\big)\big) \sim \delta\big(d\big(2\pi(k + c_{\mathfrak{g}})B - \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big)\big) \sim \delta\big(d\big(B - \frac{1}{2\pi(k+c_{\mathfrak{g}})} \sum_{j=1}^n \bar{\alpha}_j \cdot 1^{\text{shift}}_{R_j^+}\big)\big)$

The "framing" procedure mentioned above which has to be used for a rigorous treatment can also be "implemented" in the heuristic setting we work with in the present paper. This amounts to replacing (by hand) the expressions $B(\sigma^j_m)$ appearing above by[21] $\frac{1}{2}\big[B(\bar{\phi}_s(\sigma^j_m)) + B(\bar{\phi}_s^{-1}(\sigma^j_m))\big]$. Accordingly, one can expect that in the rigorous treatment where $\mathrm{WLO}(L)$ is defined and computed rigorously one has

$$\mathrm{WLO}(L) = C_1 \cdot \mathcal{S}t_{CS}(L) \tag{53}$$

where C_1 is a suitable constant independent of L (see Eq. (55) below) and where $\mathcal{S}t_{CS}(L)$ is the rigorous finite state sum (called the "Chern-Simons state sum of L in horizontal framing" in the sequel) given by

$$\mathcal{S}t_{CS}(L) := \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n \in \Lambda} \sum\nolimits_{b \in \frac{1}{k+c_{\mathfrak{g}}}\Lambda} \Big(\prod_{j=1}^n m_{\gamma_j}(\alpha_j)\Big)\Big(1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \det\nolimits_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big)$$
$$\times \prod_{j=1}^n \exp(2\pi i (\alpha_j, \sum\nolimits_m \epsilon^j_m \tfrac{1}{2}\big[B(\bar{\phi}_s(\sigma^j_m)) + B(\bar{\phi}_s^{-1}(\sigma^j_m))\big]))\Big)_{|B = b + \frac{1}{k+c_{\mathfrak{g}}}\sum_{j=1}^n \alpha_j \cdot 1^{\mathrm{shift}}_{R^+_j}} \tag{54}$$

where $s > 0$ is chosen small enough[22].

In the special case $n = 0$, i.e. the case where the link L is "empty", it follows from the heuristic definition of $\mathrm{WLO}(L)$ that we must have $\mathrm{WLO}(L) = 1$. From this and Eqs. (53), (54), and (44) we can therefore conclude

$$C_1 = \Big(\sum\nolimits_{b \in P \cap \frac{1}{k+c_{\mathfrak{g}}}\Lambda} \det\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(b))_{|\mathfrak{t}^\perp}\big)^{1-g}\Big)^{-1} \tag{55}$$

where g is the genus of Σ. In Sec. 5 below we will give a somewhat more explicit expression for C_1.

5 Equivalence of the Chern-Simons state sums and those in the shadow invariant

Theorem 5.1 *Let L be the colored link in $\Sigma \times S^1$ which we have fixed above. Then*

$$\mathcal{S}t_{CS}(L) = K^{2-2g} \cdot |X_L| \tag{56}$$

where g is the genus of Σ and

$$K := \prod_{\beta \in \mathcal{R}_+} \big(2 \sin(\tfrac{\pi(\beta,\rho)}{k+c_{\mathfrak{g}}})\big) \tag{57}$$

Before we prove Theorem 5.1 we will first introduce some notation and then state and prove two lemmas. The proof of Theorem 5.1 will be given after the proof of Lemma 2 below.

For each sequence $(\alpha_i)_{0 \le i \le n}$ of elements of Λ we set

$$B_{(\alpha_i)_i} := \tfrac{1}{k+c_{\mathfrak{g}}}\big(\alpha_0 + \sum_{j=1}^n \alpha_j \cdot 1^{\mathrm{shift}}_{R^+_j}\big).$$

Then we can rewrite Eq. (54) as

$$\mathcal{S}t_{CS}(L) = \sum_{(\alpha_i)_i \in \Lambda^{n+1}} \Big(\prod_{j=1}^n m_{\gamma_j}(\alpha_j)\Big) 1_P(B_{(\alpha_i)_i}(\sigma_0)) 1_{(\mathfrak{t}_{reg})^\Sigma}(B_{(\alpha_i)_i}) \det\nolimits_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B_{(\alpha_i)_i})_{|\mathfrak{t}^\perp})\big)$$
$$\times \prod_{j=1}^n \exp(2\pi i (\alpha_j, \sum\nolimits_m \epsilon^j_m \tfrac{1}{2}\big[B_{(\alpha_i)_i}(\bar{\phi}_s(\sigma^j_m)) + B_{(\alpha_i)_i}(\bar{\phi}_s^{-1}(\sigma^j_m))\big]\rangle\rangle \tag{58}$$

[21] in a rigorous treatment where the Hida distributions $\Psi_{\bar{\phi}_s}$, $s > 0$, are used instead of the heuristic integral functional $\int \cdots \exp(i\frac{k+c_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}(dA^\perp_c \cdot B))(DA^\perp_c \otimes DB)$ a suitably regularized version of this linear combination $\frac{1}{2}\big[B(\bar{\phi}_s(\sigma^j_m)) + B(\bar{\phi}_s^{-1}(\sigma^j_m))\big]$ appears naturally as a result of the application of the polarization identity for quadratic forms.

[22] From Lemma 1 iii) below it follows that the right-hand side of Eq. (54) as a function of s is stationary as $s \to 0$, so it is clear what "small enough" means. Moreover, Lemma 1 iii) below shows that $\mathcal{S}t_{CS}(L)$ does not depend on the special choice of the horizontal framing $(\phi_s)_{s>0}$.

Each $B_{(\alpha_i)_i}$ gives rise to an "area coloring" $\varphi_{(\alpha_i)_i} : \{Y_0, Y_1, \ldots, Y_n\} \to \Lambda$ given by

$$\varphi_{(\alpha_i)_i}(Y_t) := (k + c_{\mathfrak{g}})B_{(\alpha_i)_i}(\sigma_{Y_t}) - \rho = \alpha_0 + \sum_{j=1}^{n} \alpha_j \cdot 1^{\text{shift}}_{R_j^+}(\sigma_{Y_t}) - \rho \tag{59}$$

where σ_{Y_t} is an arbitrary point of Y_t. Note that $\rho \in \Lambda$ so $\varphi_{(\alpha_i)_i}$ is well-defined.

Lemma 1 *For each* $(\alpha_i)_{0 \le i \le n} \in \Lambda^{n+1}$ *we have*

i) $\alpha_j = \varphi_{(\alpha_i)_i}(Y_j^+) - \varphi_{(\alpha_i)_i}(Y_j^-)$ *for* $1 \le j \le n$

ii) $\det_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B_{(\alpha_i)_i})_{|\mathfrak{t}^\perp})\big) = K^{2-2g} \prod_Y (\dim(\varphi_{(\alpha_i)_i}(Y)))^{\chi(Y)}$

iii) $\prod_{j=1}^n \exp(2\pi i(\alpha_j, \sum_m \epsilon^j_m \frac{1}{2}\big[B_{(\alpha_i)_i}(\bar{\phi}_s(\sigma^j_m)) + B_{(\alpha_i)_i}(\bar{\phi}_s^{-1}(\sigma^j_m))\big])) = \prod_Y \big(v_{\varphi_{(\alpha_i)_i}(Y)}\big)^{\mathrm{gl}(Y)}$

Proof of i): By Assumption 1 the loops $l^1_\Sigma, l^2_\Sigma, \ldots, l^n_\Sigma$ do not intersect. This means that for each $j' \le n$ with $j' \neq j$ the two faces Y_j^+ and Y_j^- are either both "inside" $l^{j'}_\Sigma$ or both "outside" $l^{j'}_\Sigma$. More precisely, we have either $Y_j^+, Y_j^- \subset R^+_{j'}$ or $Y_j^+, Y_j^- \subset R^-_{j'}$. From Eq. (59) we therefore obtain $\varphi_{(\alpha_i)_i}(Y_j^+) - \varphi_{(\alpha_i)_i}(Y_j^-) = \alpha_j \cdot 1^{\text{shift}}_{R_j^+}(\sigma_{Y_j^+}) - \alpha_j \cdot 1^{\text{shift}}_{R_j^+}(\sigma_{Y_j^-}) = \alpha_j \cdot 1_{R_j^+}(\sigma_{Y_j^+}) - \alpha_j \cdot 1_{R_j^+}(\sigma_{Y_j^-}) = \alpha_j \cdot (1 - 0) = \alpha_j$.

Proof of ii): For every $b \in \mathfrak{t}$ we have

$$\begin{aligned}\det(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(b))_{|\mathfrak{t}^\perp}) &= \prod_{\beta \in \mathcal{R}_+} (1 - e^{2\pi i \beta(b)})(1 - e^{-2\pi i \beta(b)}) = \prod_{\beta \in \mathcal{R}_+} \big[-(e^{2\pi i \beta(b)/2} - e^{-2\pi i \beta(b)/2})^2\big] \\ &= \prod_{\beta \in \mathcal{R}_+} \big[-(e^{\pi i (\beta, b)} - e^{-\pi i (\beta, b)})^2\big] = \prod_{\beta \in \mathcal{R}_+} \big[-(2i \sin(\pi(\beta, b)))^2\big] = \prod_{\beta \in \mathcal{R}_+} 4 \sin(\pi(\beta, b))^2 \end{aligned} \tag{60}$$

Taking into account Eqs. (44), (59), and the relation $\sum_Y \chi(Y) = \chi(\Sigma) = 2 - 2g$ we obtain

$$\begin{aligned}\det_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B_{(\alpha_i)_i})_{|\mathfrak{t}^\perp})\big) &= \prod_{t=0}^{n} \prod_{\beta \in \mathcal{R}_+} \big(\big(2 \sin(\pi(\beta, B_{(\alpha_i)_i}(\sigma_{Y_t})))\big)^2\big)^{\chi(Y_t)/2} \\ = \prod_{t=0}^{n} \prod_{\beta \in \mathcal{R}_+} \big(2 \sin(\pi(\beta, \tfrac{1}{k+c_{\mathfrak{g}}}\big(\varphi_{(\alpha_i)_i}(Y_t) + \rho\big)))\big)^{\chi(Y_t)} &= \prod_{Y} \prod_{\beta \in \mathcal{R}_+} \big(2 \sin(\tfrac{\pi}{k+c_{\mathfrak{g}}}(\beta, \varphi_{(\alpha_i)_i}(Y) + \rho)\big)^{\chi(Y)} \\ = K^{\chi(\Sigma)} \prod_{Y} \prod_{\beta \in \mathcal{R}_+} \left(\frac{\sin(\frac{\pi(\beta, \varphi_{(\alpha_i)_i}(Y) + \rho)}{k+c_{\mathfrak{g}}})}{\sin(\frac{\pi(\beta, \rho)}{k+c_{\mathfrak{g}}})}\right)^{\chi(Y)} &= K^{2-2g} \prod_{Y} (\dim(\varphi_{(\alpha_i)_i}(Y)))^{\chi(Y)} \end{aligned} \tag{61}$$

Proof of iii): Recall that the framing $(\phi_s)_{s>0}$ was assumed to be horizontal. Thus, for fixed j and m, exactly one of the two points $\bar{\phi}_s(\sigma^j_m)$ and $\bar{\phi}_s^{-1}(\sigma^j_m)$ will lie in Y_j^+ and the other one in Y_j^- and we have for sufficiently small $s > 0$

$$B_{(\alpha_i)_i}(\bar{\phi}_s(\sigma^j_m)) + B_{(\alpha_i)_i}(\bar{\phi}_s^{-1}(\sigma^j_m)) = \tfrac{1}{k+c_{\mathfrak{g}}}(\varphi_{(\alpha_i)_i}(Y_j^+) + \varphi_{(\alpha_i)_i}(Y_j^-) + 2\rho) \tag{62}$$

Let us set

$$\epsilon_j := \sum_{m \le n^j} \epsilon^j_m = \mathrm{wind}(l^j_{S^1}) \tag{63}$$

Then, taking into account part i) of the Lemma we get (for small $s > 0$)

$$
\begin{aligned}
&\prod_j \exp(2\pi i(\alpha_j, \textstyle\sum_m \epsilon^j_m \frac{1}{2}\big[B_{(\alpha_i)_i}(\bar{\phi}_s(\sigma^j_m)) + B_{(\alpha_i)_i}(\bar{\phi}_s^{-1}(\sigma^j_m))\big]))\\
&= \prod_j \exp(2\pi i(\textstyle\sum_m \epsilon^j_m)\frac{1}{2}\frac{1}{k+c_{\mathfrak{g}}}\big(\varphi_{(\alpha_i)_i}(Y_j^+) - \varphi_{(\alpha_i)_i}(Y_j^-), \varphi_{(\alpha_i)_i}(Y_j^+) + \varphi_{(\alpha_i)_i}(Y_j^-) + 2\rho\big)\\
&= \prod_j \exp\Big(\tfrac{\pi i}{k+c_{\mathfrak{g}}}\epsilon_j\Big[(\varphi_{(\alpha_i)_i}(Y_j^+), \varphi_{(\alpha_i)_i}(Y_j^+) + 2\rho) - (\varphi_{(\alpha_i)_i}(Y_j^-), \varphi_{(\alpha_i)_i}(Y_j^-) + 2\rho)\Big]\Big)\\
&= \prod_j \exp\big(\tfrac{\pi i}{k+c_{\mathfrak{g}}}\epsilon_j\big[\mathrm{sgn}(Y_j^+; l^j_\Sigma)C_2(\varphi_{(\alpha_i)_i}(Y_j^+)) + \mathrm{sgn}(Y_j^-; l^j_\Sigma)C_2(\varphi_{(\alpha_i)_i}(Y_j^-))\big]\big)\\
&= \prod_Y \exp\Big(\tfrac{\pi i}{k+c_{\mathfrak{g}}}\Big(\textstyle\sum_{j \text{ with } \mathrm{arc}(l^j_\Sigma)\subset\partial Y} \epsilon_j\, \mathrm{sgn}(Y; l^j_\Sigma)\Big)C_2(\varphi_{(\alpha_i)_i}(Y))\Big)\\
&\overset{(*)}{=} \prod_Y \exp\Big(\tfrac{\pi i}{k+c_g}\ \mathrm{gl}(Y)\cdot C_2(\varphi_{(\alpha_i)_i}(Y))\Big) = \Big(\prod_Y \big(v_{\varphi_{(\alpha_i)_i}(Y)}\big)^{\mathrm{gl}(Y)}\Big)\cdot\big(\prod_Y e^{\frac{\pi i c}{12}\,\mathrm{gl}(Y)}\big)\\
&\overset{(**)}{=} \Big(\prod_Y \big(v_{\varphi_{(\alpha_i)_i}(Y)}\big)^{\mathrm{gl}(Y)}\Big)
\end{aligned}
$$

Here step $(*)$ follows from Eq. (23) and Eq. (63). Moreover, also step $(**)$ follows from Eq. (23) which clearly implies $\sum_Y \mathrm{gl}(Y) = 0$.

Recall that $col(X_L)$ denotes the set of mappings $\{Y_0, Y_1, \ldots, Y_n\} \to \Lambda^k_+$. In the sequel let $col'(X_L)$ denote the set of mappings $\{Y_0, Y_1, \ldots, Y_n\} \to \Lambda \cap ((k + c_{\mathfrak{g}})\mathfrak{t}_{reg} - \rho)$ and let $(\mathcal{W}_k)^{\{Y_0, Y_2, \ldots, Y_n\}}$, or simply, $(\mathcal{W}_k)^{n+1}$ denote the space of functions from $\{Y_0, Y_1, \ldots, Y_n\}$ with values in $\mathcal{W}_k$.

Lemma 2 *The mappings*

$$\Phi : \{(\alpha_i)_{0\le i\le n} \in \Lambda^{n+1} \mid 1_{P^\Sigma}(B_{(\alpha_i)_i}) \neq 0\} \ni (\alpha_i)_{0\le i\le n} \mapsto \varphi_{(\alpha_i)_i} \in col(X_L)$$

$$\Phi' : \{(\alpha_i)_{0\le i\le n} \in \Lambda^{n+1} \mid 1_{(\mathfrak{t}_{reg})^\Sigma}(B_{(\alpha_i)_i}) \neq 0\} \ni (\alpha_i)_{0\le i\le n} \mapsto \varphi_{(\alpha_i)_i} \in col'(X_L)$$

are well-defined bijections and we have

$$col'(X_L) = \{\underline{\tau}\cdot\varphi \mid \varphi \in col(X_L), \underline{\tau} \in (\mathcal{W}_k)^{n+1}\} \tag{64}$$

where $\underline{\tau}\cdot\varphi \in (\mathcal{W}_k)^{n+1}$ is given by $(\underline{\tau}\cdot\varphi)(Y) = \underline{\tau}(Y)\cdot\varphi(Y)$ for all $Y \in \{Y_0, Y_1, \ldots, Y_n\}$.

Proof.

1. Φ' *is well-defined and surjective:* Clearly, we have $\{\varphi_{(\alpha_i)_i} \mid (\alpha_i)_{0\le i\le n} \in \Lambda^{n+1}\} = \Lambda^{\{Y_0,Y_1,\ldots,Y_n\}}$. On the other hand, for fixed $(\alpha_i)_{0\le i\le n} \in \Lambda^{n+1}$ the relation $1_{(\mathfrak{t}_{reg})^\Sigma}(B_{(\alpha_i)_i}) \neq 0$ is equivalent to $\mathrm{Image}(B_{(\alpha_i)_i}) = \mathrm{Image}(\frac{1}{k+c_{\mathfrak{g}}}(\varphi_{(\alpha_i)_i} + \rho)) \subset \mathfrak{t}_{reg}$ which is equivalent to $\mathrm{Image}(\varphi_{(\alpha_i)_i}) \subset (k + c_{\mathfrak{g}})\mathfrak{t}_{reg} - \rho$. The assertion now follows.

2. Φ *is well-defined and surjective:* For fixed $(\alpha_i)_{0\le i\le n} \in \Lambda^{n+1}$ the relation $1_{P^\Sigma}(B_{(\alpha_i)_i}) \neq 0$ is equivalent to $\mathrm{Image}(B_{(\alpha_i)_i}) = \mathrm{Image}(\frac{1}{k+c_{\mathfrak{g}}}(\varphi_{(\alpha_i)_i} + \rho)) \subset P$ which is equivalent to $\mathrm{Image}(\varphi_{(\alpha_i)_i}) \subset (k + c_{\mathfrak{g}})P - \rho$. Thus the assertion follows if we can show that

$$\Lambda \cap ((k + c_{\mathfrak{g}})P - \rho) = \Lambda^k_+ \tag{65}$$

In order to prove this equation note that $P = \mathcal{C} \cap \{\lambda \in \mathfrak{t} \mid (\lambda, \theta) < 1\}$ so we have

$$\begin{aligned}
&\Lambda \cap ((k + c_{\mathfrak{g}})P - \rho) \\
&= \Lambda \cap \big(\mathcal{C} \cap \{\lambda \in \mathfrak{t} \mid (\lambda, \theta) < k + c_{\mathfrak{g}}\} - \rho\big) \\
&= \Lambda \cap (\mathcal{C} - \rho) \cap \{\lambda \in \mathfrak{t} \mid (\lambda + \rho, \theta) < k + c_{\mathfrak{g}}\} \\
&\overset{(*)}{=} \Lambda \cap \overline{\mathcal{C}} \cap \{\lambda \in \mathfrak{t} \mid (\lambda + \rho, \theta) < k + c_{\mathfrak{g}}\} \\
&= \Lambda_+ \cap \{\lambda \in \mathfrak{t} \mid (\lambda, \theta) < k + c_{\mathfrak{g}} - (\rho, \theta)\} \\
&\overset{(**)}{=} \{\lambda \in \Lambda_+ \mid (\lambda, \theta) < k + 1\} \\
&\overset{(***)}{=} \{\lambda \in \Lambda_+ \mid (\lambda, \theta) \leq k\} = \Lambda_+^k
\end{aligned}$$

Here step $(*)$ follows because for each $\alpha \in \Lambda$, $\alpha + \rho$ is in the open Weyl chamber $\mathcal{C}$ iff α is in the closure $\overline{\mathcal{C}}$, i.e. we have $\Lambda \cap (\mathcal{C} - \rho) = \Lambda \cap \overline{\mathcal{C}} = \Lambda_+$ (cf. the last remark in Sec. V.4 in [17]). Step $(**)$ follows from $c_{\mathfrak{g}} = 1 + (\theta, \rho)$ and step $(***)$ from $(\lambda, \theta) \in \mathbb{Z}$ for each $\lambda \in \Lambda$.

3. *Formula* (64) *holds:* This follows from the fact that both the mapping $\mathcal{W}_{\text{aff}} \times P \ni (\tau, b) \mapsto \tau \cdot b \in \mathfrak{t}_{reg}$ and the mapping $i : \mathcal{W}_{\text{aff}} \to \mathcal{W}_k$ in Subsec. 2.2 are bijections.

4. Φ *and* Φ' *are injective:* Let $(\alpha''_i)_i, (\alpha'_i)_i \in \{(\alpha_i)_{0 \leq i \leq n} \in \Lambda^{n+1} \mid 1_{(\mathfrak{t}_{reg})^{\Sigma}}(B_{(\alpha_i)_i}) \neq 0\}$ such that $\varphi_{(\alpha''_i)_i} = \varphi_{(\alpha'_i)_i}$. From Lemma 1 i) it then follows immediately that $\alpha''_i = \alpha'_i$ for $i \in \{1, 2, \ldots, n\}$. Moreover, from Eq. (59) and Eq. (49) we get $\alpha''_0 = \varphi_{(\alpha''_i)_i}(Y_{\sigma_0}) + \rho = \varphi_{(\alpha'_i)_i}(Y_{\sigma_0}) + \rho = \alpha'_0$ where Y_{σ_0} denotes the face which contains the point σ_0.

$\square$

Proof of Theorem 5.1: Applying Lemma 1 to Eq. (58) we obtain

$$\mathcal{S}t_{CS}(L) = K^{2-2g} \sum_{(\alpha_i)_i \in \Lambda^{n+1}} 1_P(B_{(\alpha_i)_i}(\sigma_0)) 1_{(\mathfrak{t}_{reg})^{\Sigma}}(B_{(\alpha_i)_i}) \big(\prod_{j=1}^{n} m_{\gamma_j}(\varphi_{(\alpha_i)_i}(Y_j^+) - \varphi_{(\alpha_i)_i}(Y_j^-))\big)$$
$$\times \prod_Y (\dim(\varphi_{(\alpha_i)_i}(Y)))^{\chi(Y)} \Big(\prod_Y (v_{\varphi_{(\alpha_i)_i}}(Y))^{\text{gl}(Y)}\Big) \quad (66)$$

Without loss of generality we can assume that $\sigma_0 \in Y_0$. Then we obtain from Lemma 2

$$\mathcal{S}t_{CS}(L) = K^{2-2g} \sum_{\varphi \in col'(X_L)} 1_P((k + c_{\mathfrak{g}}) \cdot (\varphi(Y_0) + \rho)) \big(\prod_{j=1}^{n} m_{\gamma_j}(\varphi(Y_j^+) - \varphi(Y_j^-))\big)$$
$$\times \prod_Y (\dim(\varphi(Y)))^{\chi(Y)} \Big(\prod_Y (v_{\varphi(Y)})^{\text{gl}(Y)}\Big) \quad (67)$$

Now observe that for all $\tau \in \mathcal{W}_k$, $b \in \Lambda \cap ((k + c_{\mathfrak{g}})\mathfrak{t}_{reg} - \rho)$ we have

$$v_{\tau \cdot b} = v_b \quad (68)$$
$$\dim(\tau \cdot b) = \text{sgn}(\tau) \dim(b) \quad (69)$$

Moreover, $1_P((k + c_{\mathfrak{g}})(\tau \cdot \varphi(Y_0) + \rho)) = 1_{\tau = 1}$ for $\varphi \in col(X_L)$, $\tau \in \mathcal{W}_k$. Thus we obtain from Eq. (67)

and Eq. (64) in Lemma 2

$$\begin{aligned}
\mathcal{S}t_{CS}(L) &= K^{2-2g} \sum_{\varphi \in col(X_L)} \sum_{\underline{\tau} \in (\mathcal{W}_k)^{n+1}} 1_{\tau_0=1} \big(\prod_{j=1}^{n} m_{\gamma_j}(\underline{\tau}(Y_j^+) \cdot \varphi(Y_j^+) - \underline{\tau}(Y_j^-) \cdot \varphi(Y_j^-) \big) \\
&\quad \times \prod_{Y} \big(\mathrm{sgn}(\underline{\tau}(Y))\big)^{\chi(Y)} \prod_{Y} (\dim(\varphi(Y)))^{\chi(Y)} \Big(\prod_{Y} (v_{\varphi(Y)})^{\mathrm{gl}(Y)} \Big) \\
&= K^{2-2g} \sum_{\varphi \in col(X_L)} \sum_{\tau_0, \tau_1, \ldots \tau_n \in \mathcal{W}_k} 1_{\tau_0=1} \prod_{t=0}^{n} \big(\mathrm{sgn}(\tau_t)\big)^{\chi(Y_t)} \big(\prod_{j=1}^{n} m_{\gamma_j}(\tau_{t(j,+)} \cdot \varphi(Y_j^+) - \tau_{t(j,-)} \cdot \varphi(Y_j^-) \big) \\
&\quad \times \prod_{Y} (\dim(\varphi(Y)))^{\chi(Y)} \Big(\prod_{Y} (v_{\varphi(Y)})^{\mathrm{gl}(Y)} \Big) \qquad (70)
\end{aligned}$$

where $t(j,+)$ resp. $t(j,-)$ is the unique index $t \in \{0, 1, 2, \ldots, n\}$ such that $Y_t = Y_j^+$ resp. $Y_t = Y_j^-$ holds. Each m_{γ_j} is invariant under the (classical) Weyl group $\mathcal{W}$. From (7) and (6) and the fact that each $\tau \in \mathcal{W}_{\mathrm{aff}}$ can be written as the product of a translation and an element of $\mathcal{W}$ it easily follows that

$$m_{\gamma_j}(\tau_{t(j,+)} \cdot \varphi(Y_j^+) - \tau_{t(j,-)} \cdot \varphi(Y_j^-) \big) = m_{\gamma_j}(\varphi(Y_j^+) - \tau_{t(j,+)}^{-1} \cdot \tau_{t(j,-)} \cdot \varphi(Y_j^-) \big) \qquad (71)$$

Accordingly, let us set $\tilde{\tau}_j := \tau_{t(j,+)}^{-1} \cdot \tau_{t(j,-)}$. Clearly, we have

$$\prod_{t=0}^{n} \big(\mathrm{sgn}(\tau_t)\big)^{\chi(Y_t)} \overset{(*)}{=} \prod_{t=0}^{n} \big(\mathrm{sgn}(\tau_t)\big)^{\#\{j \le n | \mathrm{arc}(l_\Sigma^j) \subset \partial Y_t\}} = \prod_{j=1}^{n} \mathrm{sgn}(\tau_{t(j,+)}) \, \mathrm{sgn}(\tau_{t(j,-)}) = \prod_{j=1}^{n} \mathrm{sgn}(\tilde{\tau}_j) \qquad (72)$$

(here step $(*)$ follows from $\chi(Y_t) = 2 - \#\{j \le n \mid \mathrm{arc}(l_\Sigma^j) \subset \partial Y_t\}$). On the other hand the expressions $\sum_{\tilde{\tau}_j} \mathrm{sgn}(\tilde{\tau}_j) \big(\prod_{j=1}^{n} m_{\gamma_j}(\varphi(Y_j^+) - \tilde{\tau}_j \cdot \varphi(Y_j^-)) \big)$, $j \le n$, are exactly of the form of the expression on the right-hand side of formula (17) so we have

$$\sum_{\tilde{\tau}_1, \tilde{\tau}_2, \ldots \tilde{\tau}_n \in \mathcal{W}_k} \prod_{j=1}^{n} \mathrm{sgn}(\tilde{\tau}_j) \big(\prod_{j=1}^{n} m_{\gamma_j}(\varphi(Y_j^+) - \tilde{\tau}_j \cdot \varphi(Y_j^-) \big) = \prod_{j=1}^{n} N_{\gamma_j \varphi(Y_j^+)}^{\varphi(Y_j^-)}$$

Combining this with Eqs. (70)–(72) we obtain

$$\begin{aligned}
\mathcal{S}t_{CS}(L) &= K^{2-2g} \sum_{\varphi \in col(X_L)} \big(\prod_{j=1}^{n} N_{\gamma_j \varphi(Y_j^+)}^{\varphi(Y_j^-)} \big) \prod_{Y} (\dim(\varphi(Y)))^{\chi(Y)} \Big(\prod_{Y} (v_{\varphi(Y)})^{\mathrm{gl}(Y)} \Big) \\
&= K^{2-2g} \sum_{\varphi \in \mathrm{col}(X_L)} |X_L|_1^{\varphi} |X_L|_2^{\varphi} |X_L|_3^{\varphi} \\
&= K^{2-2g} |X_L| \qquad \textbf{q.e.d.}
\end{aligned}$$

From Theorem 5.1 and Eq. (53) above we can conclude that WLO(L) coincides with $|X_L|$ up to a multiplicative constant (independent of L). We can easily determine this multiplicative constant explicitly. According to Eqs. (55), (60), and Eq. (12) we have (cf. Eq. (65) and Example 3 above)

$$C_1 = \frac{1}{\sum_{\lambda \in \Lambda_+^k} (K \dim(\lambda))^{2-2g}} = \frac{1}{K^{2-2g}} \frac{1}{|X_\emptyset|} \qquad (73)$$

so from Eq. (53) and Theorem 5.1 we finally obtain

$$\mathrm{WLO}(L) = \frac{|X_L|}{|X_\emptyset|} \qquad (74)$$

This agrees exactly with the formula appearing at the end of Subsec. 4.1 above.

6 Outlook

In the introduction we mentioned one of the most important open questions in the theory of 3-manifold quantum invariants, the question whether and how one can make rigorous sense of Witten's heuristic path integral expressions for the Wilson loop observables of Chern-Simons theory, cf. the r.h.s. of Eq. (34). A related and probably less difficult question is whether and how one can make rigorous sense of those path integral expressions that arise from the r.h.s. of Eq. (34) after choosing a suitable gauge fixing. Until recently Lorentz gauge fixing was the only[23] gauge fixing procedure for which the relevant path integral expressions have been evaluated completely for general groups, links and manifolds, cf. [29, 9, 10, 7, 16, 8, 4]. The final result of this evaluation is a complicated infinite series whose terms involve integrals over (high-dimensional) "configuration spaces", cf. [16, 4]. The heuristic path integral expressions which appear during the intermediate computations are even more complicated and it should be very hard to find a rigorous realization of these path integrals[24].

It is therefore desirable to find other gauges for which the WLOs can also be evaluated explicitly. A gauge which leads to the expressions appearing in Turaev's shadow world approach to the 3-manifold quantum invariants would be particularly desirable. This is because the expressions appearing in the shadow world approach involve only finite sums, which are defined in a purely combinatorial way. These finite combinatorial sums are considerably less complicated than the infinite series of configuration space integrals mentioned above. Accordingly, it is reasonable to believe that for such a gauge fixing also the corresponding path integral expressions and the heuristic arguments used for their evaluation will be less complicated than those for Lorentz gauge fixing.

The results in [32] and the present paper suggest that for manifolds M of the form $M = \Sigma \times S^1$ torus gauge fixing is a gauge fixing with the desired properties. Moreover, as explained in Subsec. 4.3 above (and in more detail in Secs. 8–9 in [31] and Sec. 4–6 in [32]) one can hope that the path integral expressions for the WLOs in the torus gauge, i.e. the r.h.s. of Eq. (42) above, are indeed simple enough to admit a rigorous treatment. In fact, all of the heuristic integrations appearing in Eq. (42) are of "Gaussian type" and with the help of suitable regularization techniques it should be possible to find a rigorous realization.

The crucial remaining question, currently studied in [34], is whether the computations in Subsec. 5.2 in [32] and Subsec. 4.4 of the present paper can be generalized to general links, i.e. links with double points, and whether these computations will lead to the r.h.s. of Eq. (18) above also for these general links (for which $|X_L|_4^{\varphi}$ is non-trivial). If it turns out that this question has a positive answer then this will probably settle the second of the two open questions mentioned above. Moreover, one can then hope to make some progress regarding a rigorous realization of the r.h.s. of Eq. (34) *before* a gauge fixing is applied.

7 Appendix: A path integral derivation of the quantum Racah formula

In [12] WLO(L) was evaluated in the torus gauge approach in the special case where the link L consists exclusively of 3 vertical loops with colors $\lambda, \mu, \nu \in \Lambda_+^k$ (cf. Remark 5 below). The result of this evaluation is the expression on the right-hand side of Eq. (13) above. As we showed in Secs. 4–5 when evaluating the WLOs of loops without double points in the torus gauge approach the expressions on the right-hand side of Eq. (17) arise naturally. In other words: both the left-hand side and the right-hand side of Eq. (17) appear naturally in the torus gauge approach when computing the WLOs of suitable links. One can therefore hope to obtain a path integral derivation of Eq. (17) by considering links that contain both vertical loops and loops without double points. Accordingly, let us now generalize some of the results obtained in Secs. 4 and 5 to this more general situation where the (colored) link $L = ((l_1, l_2, \ldots, l_N), (\gamma_1, \gamma_2, \ldots, \gamma_N))$ is allowed to contain vertical loops. More precisely, we assume that

[23] for CS models on the special manifold $M = \mathbb{R}^3$ there is an alternative approach based on light-cone gauge fixing and a suitable complexification of the manifold, cf. [26], [38]. However, in this approach certain correction factors have to be inserted "by hand" in the course of the computations. At present the origin of these correction factors is not clear

[24] There are, however, some interesting partial results in this direction, cf. [3]

the sub link $(l_1, l_2, \ldots, l_n)$, $n \le N$, is admissible and each loop l_k for $k \in \{n+1, \ldots, N\}$ is a "vertical" loop "above" the point $\sigma_k \in \Sigma$, i.e. l^k_Σ is a constant mapping taking only the value σ_k. Moreover, we will assume for simplicity that $\mathrm{wind}(l^k_{S^1}) = 1$ for each $k \in \{n+1, \ldots, N\}$.

Then, using similar arguments as in Subsec. 4.4 we can again derive Eq. (53) where $\mathcal{S}t_{CS}(L)$ is now given by

$$\begin{aligned}\mathcal{S}t_{CS}(L) := \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in\Lambda} \sum\nolimits_{b\in\frac{1}{k+c_{\mathfrak{g}}}\Lambda} (\prod_{j=1}^{n} m_{\gamma_j}(\alpha_j)) \Big(1_{(\mathfrak{t}_{reg})^\Sigma}(B) 1_P(B(\sigma_0)) \det\nolimits_{reg}\big(1_{\mathfrak{t}^\perp} - \exp(\mathrm{ad}(B)_{|\mathfrak{t}^\perp})\big) \\ \times (\prod_{k=n+1}^{N} \chi_{\gamma_k}(\exp(B(\sigma_k)))) \\ \times \prod_{j=1}^{n} \exp(2\pi i(\alpha_j, \sum\nolimits_k \epsilon^j_k \tfrac{1}{2}\big[B(\bar{\phi}_s(\sigma^j_k)) + B(\bar{\phi}_s^{-1}(\sigma^j_k))\big])) \Big)_{|B = b + \frac{1}{k+c_{\mathfrak{g}}} \sum_{j=1}^n \alpha_j \cdot 1^{\mathrm{shift}}_{R^+_j}}\end{aligned} \tag{75}$$

for sufficiently small $s > 0$. Recall that χ_{γ_k} is the character associated to the dominant weight γ_k (not to confused with the notation $\chi(Y_t)$ for the Euler characteristic of the face Y_t).

Also the computations in the proof of Theorem 5.1 can be generalized in a straightforward way. One obtains

$$\begin{aligned}\mathcal{S}t_{CS}(L) = K^{2-2g} \sum_{\varphi\in col(X_L)} (\prod_{j=1}^{n} M^{\varphi(Y_j^-)}_{\gamma_j \varphi(Y_j^+)}) (\prod_{k=n+1}^{N} \chi_{\gamma_k}(\exp(\tfrac{1}{k+c_{\mathfrak{g}}}(\varphi(Y_{\sigma_k}) + \rho)))) \\ \times \prod_Y (\dim(\varphi(Y)))^{\chi(Y)} \Big(\prod_Y (v_{\varphi(Y)})^{\mathrm{gl}(Y)}\Big)\end{aligned} \tag{76}$$

where Y_{σ_k}, $k \in \{n+1, \ldots, N\}$, denotes the face in which σ_k lies and where we have set

$$M^\beta_{\gamma\alpha} := \sum_{\tau\in\mathcal{W}_k} \mathrm{sgn}(\tau) m_\gamma(\alpha - \tau(\beta)) \tag{77}$$

(According to Eq. (17) we have $M^\beta_{\gamma\alpha} = N^\beta_{\gamma\alpha}$ so we could replace $M^\beta_{\gamma\alpha}$ by $N^\beta_{\gamma\alpha}$ above. But as we want to give a path integral derivation of Eq. (17) which is based on Eq. (76) we avoid this replacement here.) Now observe that

$$\chi_\mu(\exp(\tfrac{1}{k+c_{\mathfrak{g}}}(\lambda + \rho))) = \frac{S_{\mu\lambda}}{S_{0\lambda}} \tag{78}$$

Eq. (78) follows from the definition of the S-matrix in Subsec. 2.2 if one takes into account Weyl's character formula[25]. Combining Eqs. (53), (73) (76), and (78) we finally obtain

$$\begin{aligned}&\mathrm{WLO}(L) \\ &= \frac{1}{|X_\emptyset|} \sum_{\varphi\in col(X_L)} (\prod_{j=1}^{n} M^{\varphi(Y_j^-)}_{\gamma_j \varphi(Y_j^+)}) (\prod_{k=n+1}^{N} \frac{S_{\gamma_k \varphi(Y_{\sigma_k})}}{S_{0\varphi(Y_{\sigma_k})}}) \prod_Y (\dim(\varphi(Y)))^{\chi(Y)} \Big(\prod_Y (v_{\varphi(Y)})^{\mathrm{gl}(Y)}\Big)\end{aligned} \tag{79}$$

In the special case $n = 0$, i.e. in the case where there are only vertical loops, there is only one face $Y_0 = \Sigma$ and we have $\mathrm{gl}(Y_0) = 0$, $\chi(Y_0) = \chi(\Sigma) = 2 - 2g$ so Eq. (79) then reduces to

$$\mathrm{WLO}(L) = \frac{1}{|X_\emptyset|} \sum_{\lambda\in\Lambda^k_+} (\prod_{k=1}^{N} \frac{S_{\gamma_k\lambda}}{S_{0\lambda}}) \dim(\lambda)^{2-2g} \tag{80}$$

[25] i.e. formula $\chi_\mu(\exp(b)) = \frac{\sum_{w\in\mathcal{W}} \mathrm{sgn}(w) e^{2\pi i(\mu+\rho, w\cdot b)}}{\sum_{w\in\mathcal{W}} \mathrm{sgn}(w) e^{2\pi i(\rho, w\cdot b)}}$ for every $b \in \mathfrak{t}$, cf. e.g., [17] Chap. VI, Theorem 1.7, part (i) and part (iii)

Remark 5 In the special case $G = SU(2)$ the last equation is equivalent to formula (7.27) in [12]. We remark that for vertical loops the inner integral in Eq. (42) is trivial, so for the derivation of Eq. (80) one does not need the general formula (42) but can work with the simpler formulas appearing in [12], cf. equations (7.1) and (7.24) in [12]. In the special case where $N = 3$ and $\Sigma = S^2$, i.e. $g = 0$ we get from Eq. (80) (setting $\lambda := \gamma_1$, $\mu := \gamma_2$, $\nu := \gamma_3$)

$$\mathrm{WLO}(L) = \frac{1}{|X_\emptyset|} \sum_{\lambda_0} \frac{S_{\lambda\lambda_0}}{S_{0\lambda_0}} \frac{S_{\mu\lambda_0}}{S_{0\lambda_0}} \frac{S_{\nu\lambda_0}}{S_{0\lambda_0}} \dim(\lambda_0)^2 = \frac{N_{\lambda\mu\nu}}{\sum_{\lambda_0} S^2_{0\lambda_0}} = N_{\lambda\mu\nu} \tag{81}$$

(here we have used $\sum_{\lambda_0} S^2_{0\lambda_0} = (S \cdot S^T)_{00} = (S^2)_{00} = C_{00} = 1$). By combining Eq. (81) with Eq. (4.36) in [53] one obtains[26] the fusion rules (15), cf. Sec. 7.6 in [12]. (Observe that the expression N_{ijk} in Eq. (4.36) in [53] does not correspond to $N_{ijk} = N^{i^*}_{jk}$ in our notation but to $\hat{N}^{i^*}_{jk}$, cf. Remark 2).

In order to obtain a path integral derivation of Eq. (17) let us now consider a link L in $\Sigma = S^2$ which consists of 2 vertical loops l_2, l_3 over the point σ_2 resp. σ_3 with colors μ and ν and one non-vertical loop l_1 with color λ. We assume that $\mathrm{wind}(l^i_{S^1}) = 1$ for all $i = 1, 2, 3$ and that l^1_Σ is a Jordan loop (i.e. a simple loop without crossings). Moreover, we assume that σ_2, σ_3 are on different sides of l^1_Σ, i.e. that the loop projections $l^1_\Sigma, l^2_\Sigma, l^3_\Sigma$ look as in Fig. 4

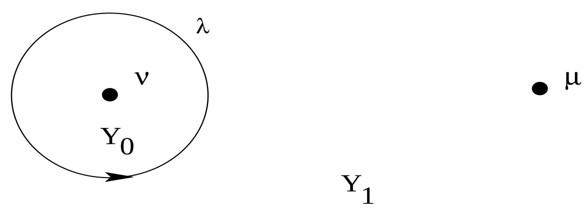

Figure 4:

We will now evaluate $\mathrm{WLO}(L)$ in two different ways. By comparing these two different evaluations of $\mathrm{WLO}(L)$ with each other we will then obtain a system of linear equations for the "unknowns" $M^\beta_{\gamma\alpha}$, $\alpha, \beta, \gamma \in \Lambda^k_+$. Later we will show that the unique solution of this system of linear equations is $M^\beta_{\gamma\alpha} = N^\beta_{\gamma\alpha}$, $\alpha, \beta, \gamma \in \Lambda^k_+$, which is nothing but formula (17).

1. Evaluation: Let us apply Eq. (79) to the link L. Observe that we have to take $n = 1$, $N = 3$ and $(\gamma_1, \gamma_2, \gamma_3) = (\lambda, \mu, \nu)$. We obtain

$$\mathrm{WLO}(L) = \frac{1}{|X_\emptyset|} \sum_{\varphi \in col(X_L)} M^{\varphi(Y_1^-)}_{\lambda\varphi(Y_1^+)} \Big(\prod_{k=2}^{3} \frac{S_{\gamma_k \varphi(Y_{\sigma_k})}}{S_{0\varphi(Y_{\sigma_k})}}\Big) \prod_{t=0}^{1} (\dim(\varphi(Y_t)))^{\chi(Y_t)} \left(\prod_{t=0}^{1} (v_{\varphi(Y_t)})^{\mathrm{gl}(Y_t)}\right) \tag{82}$$

Note that in this situation there are only two faces namely $Y_0 := Y_1^+$ and $Y_1 := Y_1^-$ and we have $\sigma_2 \in Y_0$ and $\sigma_3 \in Y_1$ (cf. Fig. 4, note that the point σ_2 is labelled by the letter ν and σ_3 by the letter μ). Moreover, $\mathrm{gl}(Y_0) = 1$, $\mathrm{gl}(Y_1) = -1$, $\chi(Y_1) = 1$, $\chi(Y_0) = 2 - 2g - 1 = 1$ (as $\Sigma = S^2$, so $g = 0$). Using the variables $\lambda_0 := \varphi(Y_0)$ and $\lambda_1 := \varphi(Y_1)$ we now obtain

$$\begin{aligned} \mathrm{WLO}(L) &= \frac{1}{|X_\emptyset|} \sum_{\lambda_0, \lambda_1} M^{\lambda_1}_{\lambda\lambda_0} \frac{S_{\nu\lambda_0}}{S_{0\lambda_0}} \frac{S_{\mu\lambda_1}}{S_{0\lambda_1}} \dim(\lambda_0) \dim(\lambda_1) T_{\lambda_0\lambda_0} T^{-1}_{\lambda_1\lambda_1} \\ &= \frac{1}{|X_\emptyset|} \frac{1}{S^2_{00}} \sum_{\lambda_0, \lambda_1} M^{\lambda_1}_{\lambda\lambda_0} S_{\nu\lambda_0} S_{\mu\lambda_1} T_{\lambda_0\lambda_0} T^{-1}_{\lambda_1\lambda_1} \end{aligned} \tag{83}$$

2. Evaluation: Let us now consider for a while a colored link $\hat{L} = ((\hat{l}_1, \hat{l}_2, \hat{l}_3), (\lambda, \mu, \nu))$ in $\Sigma = S^2$ where each $\hat{l}_j$, $j \in \{1, 2, 3\}$, is a vertical loop over the point σ_j with $\mathrm{wind}(\hat{l}^j_{S^1}) = 1$, cf. Fig.5. From Eq. (80)

[26] Note that, strictly speaking, this is not quite a "path integral derivation" of the fusion rules since the derivation of the Eq. (4.36) in [53] is not based solely on the CS path integral. In fact, since the numbers $\hat{N}^i_{jk}$ are defined abstractly, a genuine path integral derivation of the fusion rules (15) can not be expected.

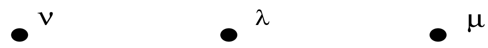

Figure 5:

(or Eq. (81)) we obtain $\mathrm{WLO}(\hat{L}) = \frac{1}{|X_\emptyset|}\frac{1}{S_{00}^2} N_{\lambda\mu\nu}$. By carrying out two simple surgery operations[27] the link $\hat{L}$ can be transformed into (a link which is isotopic to) the link L. According to Sec 4.5 in [53] each of these two surgery operations alters the value of the WLO by a factor $\frac{T_{\mu\mu}}{T_{00}}$ and $\frac{T_{\nu\nu}}{T_{00}}^{-1}$ (cf. Remark 6 below), i.e. we have

$$\mathrm{WLO}(L) = \frac{T_{\mu\mu}}{T_{\nu\nu}} \mathrm{WLO}(\hat{L}) = \frac{1}{|X_\emptyset|} \frac{1}{S_{00}^2} \frac{T_{\mu\mu}}{T_{\nu\nu}} N_{\lambda\mu\nu} \tag{84}$$

Conclusion: By combining the two equations (83) and (84) we obtain

$$\frac{T_{\mu\mu}}{T_{\nu\nu}} N_{\lambda\mu\nu} = \sum_{\lambda_0,\lambda_1} M_{\lambda\lambda_0}^{\lambda_1} \; S_{\nu\lambda_0} S_{\mu\lambda_1} T_{\lambda_0\lambda_0} T_{\lambda_1\lambda_1}^{-1} \tag{85}$$

On the other hand according to Eq. (30) in Example 2 above we have

$$\frac{T_{\mu\mu}}{T_{\nu\nu}} N_{\lambda\mu\nu} = \sum_{\lambda_0,\lambda_1} N_{\lambda\lambda_0}^{\lambda_1} \; S_{\nu\lambda_0} S_{\mu\lambda_1} T_{\lambda_0\lambda_0} T_{\lambda_1\lambda_1}^{-1} \tag{86}$$

Eq. (85) and Eq. (86) imply $\sum_{\lambda_0,\lambda_1} M_{\lambda\lambda_0}^{\lambda_1} \; S_{\nu\lambda_0} S_{\mu\lambda_1} T_{\lambda_0\lambda_0} T_{\lambda_1\lambda_1}^{-1} = \sum_{\lambda_0,\lambda_1} N_{\lambda\lambda_0}^{\lambda_1} \; S_{\nu\lambda_0} S_{\mu\lambda_1} T_{\lambda_0\lambda_0} T_{\lambda_1\lambda_1}^{-1}$. This holds for arbitrary $\lambda, \mu, \nu \in \Lambda_+^k$ so using the fact that S-matrix and the T-matrix are invertible (cf. Eqs. (11) above) we indeed obtain $M_{\mu\nu}^{\lambda} = N_{\mu\nu}^{\lambda}$ (for all $\lambda, \mu, \nu \in \Lambda_+^k$).

Remark 6 Witten's argument from Sec 4.5 in [53] which we used in the paragraph preceding Eq. (84) is based on ideas from conformal field theory. So if we want to give a (complete) path integral derivation of Eq. (17) we will have to derive the first equality in Eq. (84) using only path integral methods. It is shown in [35] how this can be done. On the other hand, if one is happy with "mixing" arguments from conformal field theory and arguments based on the CS path integral then the derivation of the (elementary) quantum Racah formula Eq. (17) which we have just given is fine and by combining Eq. (17) with the fusion rules derived in Remark 5 (using Eq. (4.36) in [53]) one finally obtains $\hat{N}_{\gamma\alpha}^{\beta} = \sum_{\tau\in\mathcal{W}_k} \mathrm{sgn}(\tau) m_\gamma(\alpha - \tau(\beta))$. Clearly, this is the first of the two versions of the "abstract" quantum Racah formula appearing at the end of Subsec. 2.2.

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[27] the first surgery operation involves a suitable tubular neighborhood (cf. Sec. 4.2 in [53]) of the loop $\hat{l}_3$, i.e. the vertical loop represented by the point "ν" in Fig. 5; the second surgery operations involves a similar tubular neighborhood of the loop $\hat{l}_2$

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